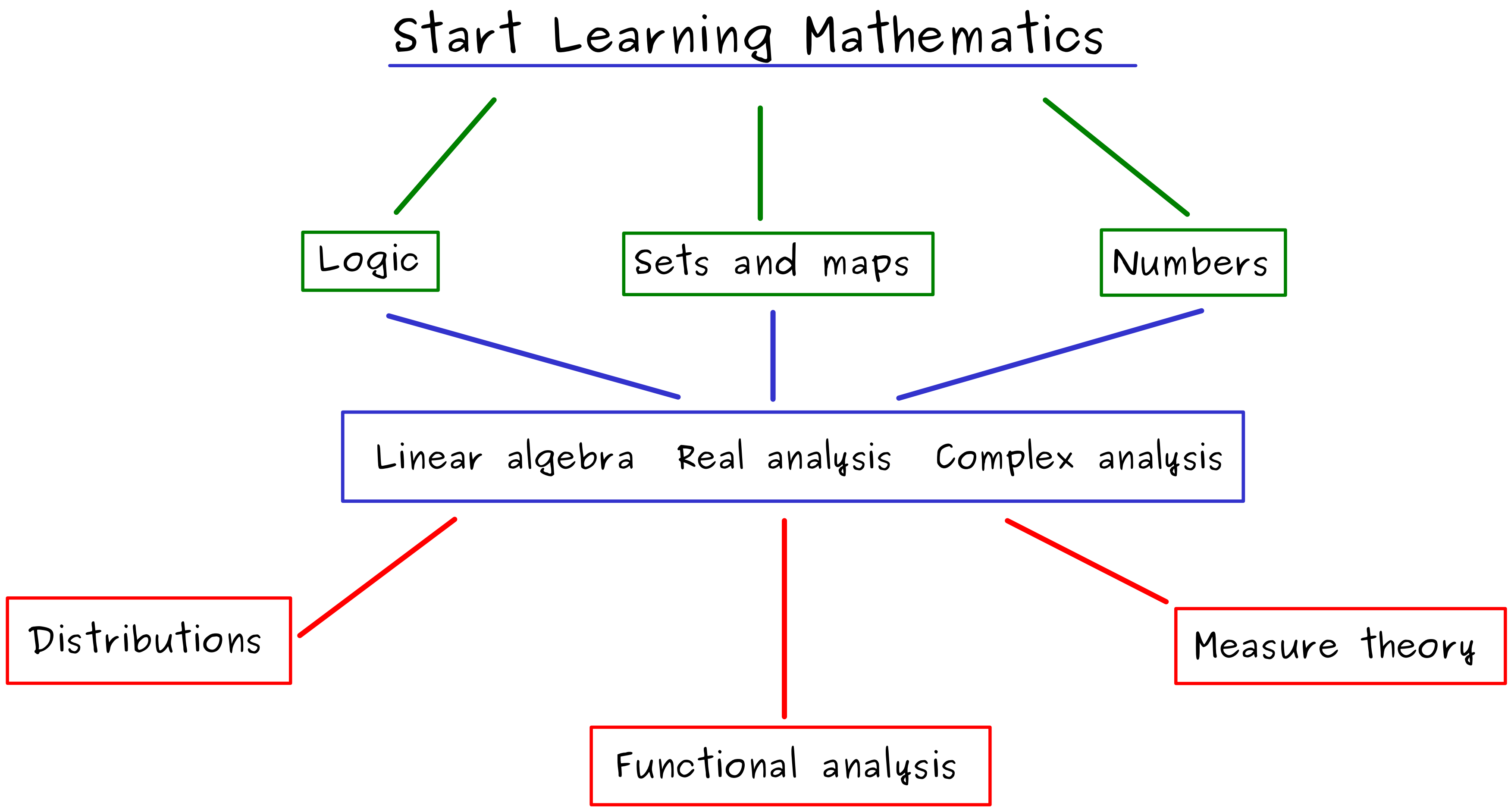
The Bright Side of Mathematics

The following pages cover the whole Start Learning Mathematics course of the Bright Side of Mathematics. Please note that the creator lives from generous supporters and would be very happy about a donation. See more here: https://tbsom.de/support

Have fun learning mathematics!


Start Learning Mathematics
Logic
Sets and maps
Numbers
Linear algebra
Real analysis
Complex analysis
Distributions
Functional analysis
Measure theory

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Start Learning Logic – Part 1

Logical statement (proposition): Statement that is either True or False

Examples:

(a) Mars is a planet (True logical statement)

(b) Pluto is a planet (False logical statement)

(c) $1 + 1 = 2$ (True logical statement)

(d) The number 5 is smaller than the number 2 (False logical statement)

(e) Good morning! (Not a logical statement)

(f) $x + 1 = 1$ (Not a logical statement) $\rightsquigarrow$ predicate

Logical operations:

Negation: For a logical statement A,
$\neg A$ denotes the negation.

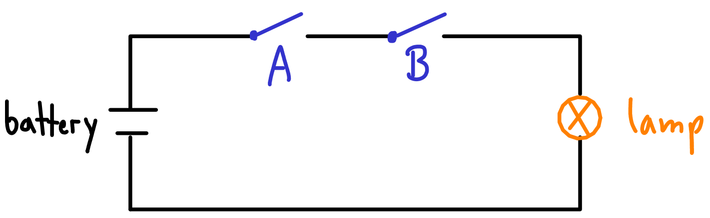
Truth table

A	$\neg A$
T	F
F	T

Examples: (a) A = The wine bottle is full
$\neg A$ = The wine bottle is not full

(b) $A = 2 + 2 = 5$
$\neg A = 2 + 2 \neq 5$

Conjunction: For two logical statements A, B,
$A \wedge B$ denotes the conjuction.



Truth table

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

Start Learning Logic - Part 2

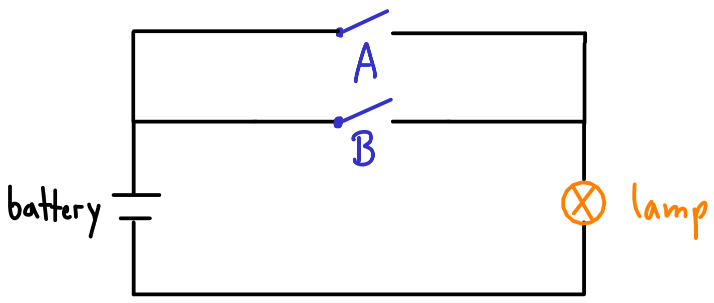
Logical statements A, B $\rightsquigarrow$ new logical statements $\neg A$, $A \wedge B$

logical variables

logical operations

Logical operations:

Disjunction: For two logical statements A, B, $A \vee B$ denotes the disjunction.



Truth table

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

Example: $\neg A \vee A$

Truth table

A	$\neg A$	$\neg A \vee A$
T	F	T
F	T	T

We say $\neg A \vee A$ is a tautology.

always true
(independent of the truth values of the logical variables that are contained)

Logical equivalence:

Two logical statements are called logically equivalent if the truth tables (all possible assignments of truth values for the logical variables) are the same.

Example: $\neg(A \vee B) \Longleftrightarrow (\neg A) \wedge (\neg B)$

A	B	$A \vee B$	$\neg A$	$\neg B$	$\neg(A \vee B)$	$\neg A \wedge \neg B$
T	T	T	F	F	F	F
T	F	T	F	T	F	F
F	T	T	T	F	F	F
F	F	F	T	T	T	T

Start Learning Logic - Part 3

Logical operations:

Conditional: For two logical statements A, B,
$A \rightarrow B$ denotes the conditional.

Truth table

A	B	$A \rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

$\Rightarrow$ means $\rightarrow$ gives tautology

A	B	$A \wedge B$	$A \wedge B \rightarrow B$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

tautology

We can write:

$$A \wedge B \Rightarrow B$$

Biconditional: For two logical statements A, B,
$A \leftrightarrow B$ denotes the biconditional.

$\Leftrightarrow$ means $\leftrightarrow$ gives tautology

Truth table

A	B	$A \leftrightarrow B$
T	T	T
T	F	F
F	T	F
F	F	T

Example: (a) $A \leftrightarrow B \quad \Leftrightarrow \quad (A \rightarrow B) \wedge (B \rightarrow A)$

(b) $A \rightarrow B \quad \Leftrightarrow \quad \neg B \rightarrow \neg A$ (contraposition)

If there is fog, then we have poor visibility

If we don't have poor visibility, there is no fog.

Deduction rules: (how to get new true propositions from other true propositions)

Modus ponens: If $A \to B$ true and A true, then: B true

A	B	$A \to B$
T	T	T
T	F	F
F	T	T
F	F	T

Chain syllogism: If $A \to B$ true and $B \to C$ true, then: $A \to C$ true

Reductio ad absurdum: If $A \to B$ true and $A \to \neg B$ true, then: $\neg A$ true

The Bright Side of Mathematics

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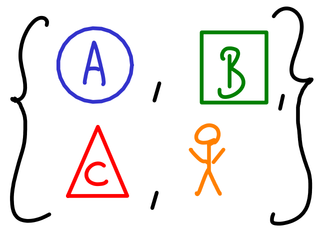
Start Learning Sets – Part 1

Propositional Logic + Naive set theory		Logic + Axioms of set theory
⇓		⇓
Goal: $\boxed{\text{doing mathematics}}$	⟵	foundation of mathematics

Set: Collection of distinct objects into a whole



set M

Such an object x inside a set M is called an element of M, write: $x \in M$.

$A \in M$

If x is not such an object inside the set M, we write: $x \notin M$ ← means: $\neg(x \in M)$

$D \notin M$

A set can be defined by giving all its elements: $A := \{2, 5, 6\}$ ($:=$ defined by)

Examples:

- Empty set: $\emptyset := \{\}$
- Natural numbers: $\mathbb{N} := \{1, 2, 3, 4, 5, \dots\}$
- Natural numbers (including zero): $\mathbb{N}_0 := \{0, 1, 2, 3, 4, \dots\}$
- Integers: $\mathbb{Z} := \{\dots, -2, -1, 0, 1, 2, \dots\}$
- Rational numbers $\mathbb{Q}$ Real numbers $\mathbb{R}$ Complex numbers $\mathbb{C}$

⇝ quantifiers $\forall\ \exists$ predicates $x \in \mathbb{N}$

Start Learning Sets – Part 2

$\boxed{1}$ is an even number — false logical statement

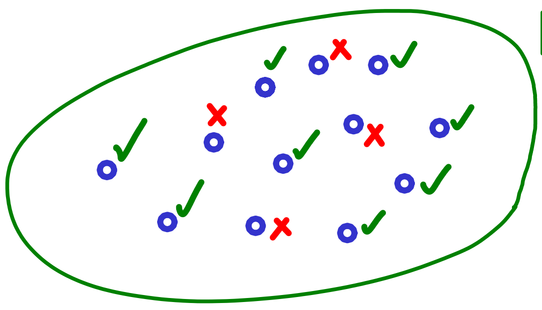
$\boxed{1}$ is an animal — false logical statement

$\underbrace{\boxed{1} + 8 = 9}_{\text{predicates}}$ — true logical statement

Predicate: An expression with undetermined variables that ascribes a property to objects filled in for the variables.

Form new sets:

$$\{ x \in \mathbb{N} \mid x \text{ is an even number} \}$$



$\mathbb{N}$

$$\{ y \in \mathbb{Z} \mid y \in \mathbb{N} \}$$

For $A := \{$Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, Neptune$\}$

form: $\{ p \in A \mid p \text{ has at least 1 confirmed moon} \}$

Quantifiers: $\forall x$ for all x $\qquad$ $\exists x$ it exists x

Predicate: x is a planet

$\underbrace{\forall x : x \text{ is a planet}}_{\text{false}} \rightsquigarrow$ logical statement

$\underbrace{\exists x : x \text{ is a planet}}_{\text{true}} \rightsquigarrow$ logical statement

Equality for sets: Two sets A, B are the same, written as $A = B$ if

$$\forall x : \quad x \in A \leftrightarrow x \in B \qquad \text{is true.}$$

Example: $C := \{2, 3, 5\} = \{3, 5, 2\} =: D$

$1 \in C \leftrightarrow 1 \in D$ true

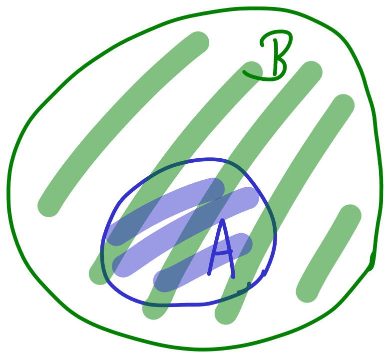
$2 \in C \leftrightarrow 2 \in D$ true

$\vdots$

$$\{2, 3, 5\} = \{2, 2, 2, 3, 3, 5\}$$

Subsets: For two sets A, B, we write $A \subseteq B$ if

$$\underbrace{\forall x : \quad x \in A \rightarrow x \in B}_{\text{Short notation: } \forall x \in A : \ x \in B} \qquad \text{is true.}$$



We call A a subset of B. (We can also write $B \supseteq A$)

Start Learning Sets - Part 3

$A \subseteq B$ ← is a superset of A

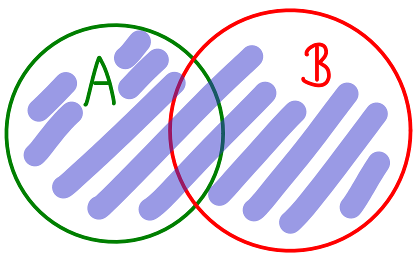
↖ is a subset of B

$\rightsquigarrow$ $B \subseteq B$ ✓

$\emptyset \subseteq B$ ✓

$\forall x : x \in \emptyset \rightarrow x \in B$

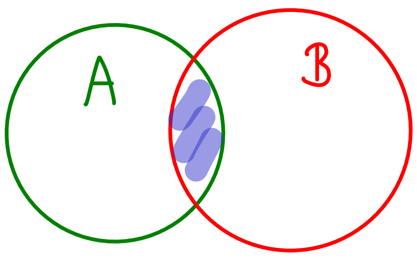
Union:



$$A \cup B := \{ x \mid x \in A \vee x \in B \}$$

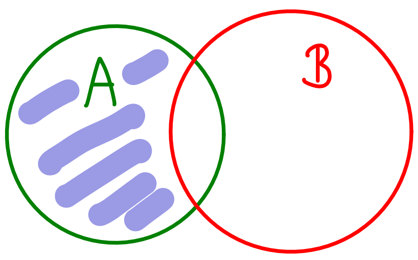
$(\forall x : \; x \in A \cup B \leftrightarrow x \in A \vee x \in B)$ is true

Intersection:



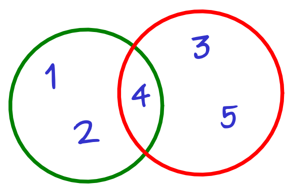
$$A \cap B := \{ x \mid x \in A \wedge x \in B \}$$

Set difference:



$$A \setminus B := \{ x \mid x \in A \wedge x \notin B \}$$

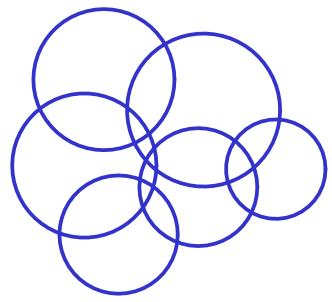
Example: $A := \{1, 2, 4\}$, $B := \{3, 4, 5\}$



$A \cup B = \{1, 2, 3, 4, 5\}$, $A \cap B = \{4\}$, $A \setminus B = \{1, 2\}$

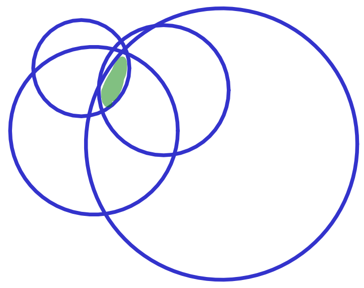
Big union: Need: I set, A_i set for each $i \in I$.

$$\bigcup_{i \in I} A_i := \{ x \mid \exists i \in I : x \in A_i \}$$



Big intersection:

$$\bigcap_{i \in I} A_i := \{ x \mid \forall i \in I : x \in A_i \}$$



Example: $A_1 = \{1\}$, $A_2 = \{2\}$, $A_3 = \{3\}$, ...

$I = \mathbb{N}$, $A_i = \{i\}$. Then: $\bigcup_{i \in I} A_i = \{1, 2, 3, \dots\} = \mathbb{N}$

$$\bigcap_{i \in I} A_i = \emptyset$$

Power set: For a set A define $\mathcal{P}(A) := \{ X \mid X \subseteq A \}$ The set of all subsets of A

Example: $A = \{1, 2, 3\}$, $\mathcal{P}(A) = \{ \emptyset, \{1,2,3\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\} \}$

Number of elements: $|A| = 3$, $|\mathcal{P}(A)| = 8 = 2^3$

Start Learning Sets – Part 4

Cartesian product: $A \times B$ set of all ordered pairs

$A := \{\Delta, \square, \bigcirc\}$

$B := \{4, 7\}$

$\rightsquigarrow (\Delta, 7)$

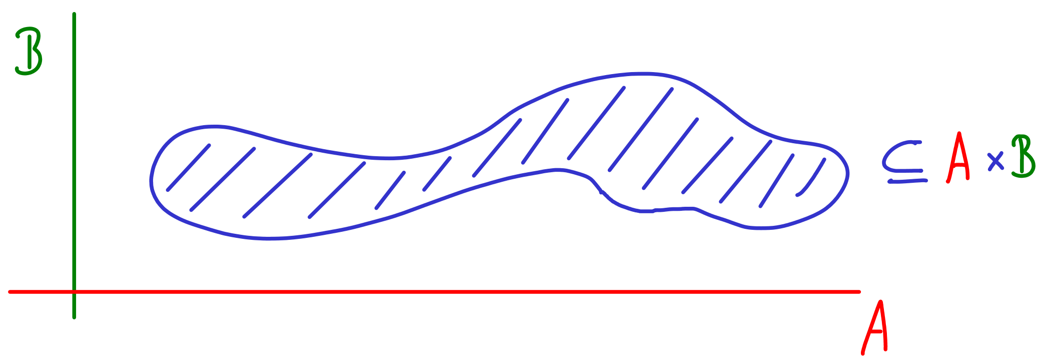
7	$(\Delta, 7)$	$(\square, 7)$	$(\bigcirc, 7)$
4	$(\Delta, 4)$	$(\square, 4)$	$(\bigcirc, 4)$
	Δ	$\square$	$\bigcirc$

Definition of ordered pair: For elements x, y write $(x, y) := \{ \{x\}, \{x, y\} \}$

$$(x, y) = (\tilde{x}, \tilde{y}) \iff \{x\} = \{\tilde{x}\} \wedge \{x, y\} = \{\tilde{x}, \tilde{y}\}$$

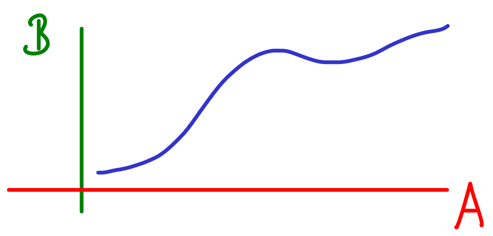
$$\iff x = \tilde{x} \wedge y = \tilde{y}$$

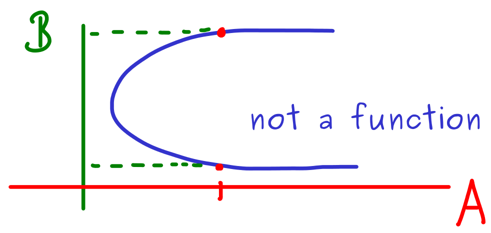
Definition: $A \times B := \{ (a, b) \mid a \in A \wedge b \in B \}$



A subset $G_f \subseteq A \times B$ is called a function if

$$(\forall x \; \forall y \; \forall \tilde{y} : (x, y) \in G_f \wedge (x, \tilde{y}) \in G_f \rightarrow y = \tilde{y}) \text{ is true.}$$





If also $\forall x \in A : \exists y \in B : (x,y) \in G_f$ is true,

we write: $f : A \to B$ and $f(x) = y$ for $(x,y) \in G_f$

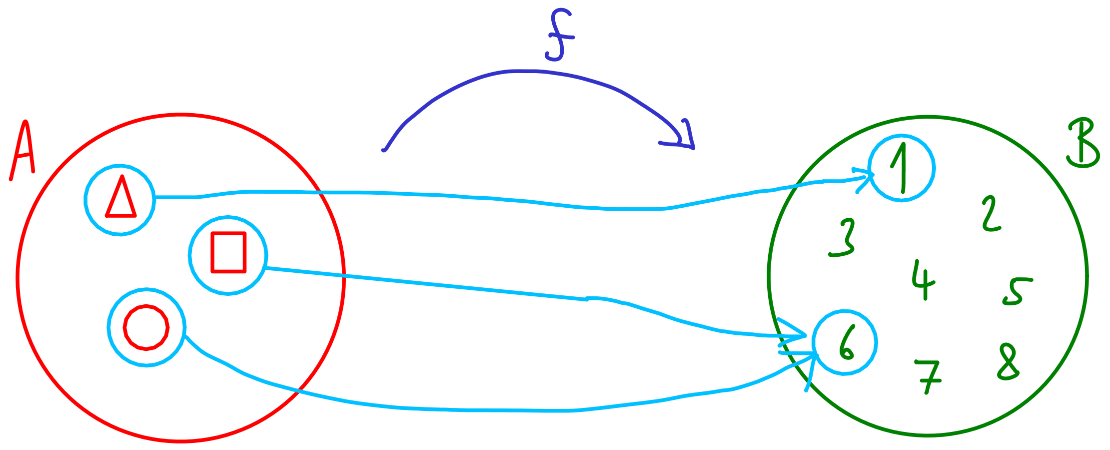
codomain of f

domain of f

a map from A into B

graph of f

Example:



$f(\Delta) = 1$

$f(\bigcirc) = 6$

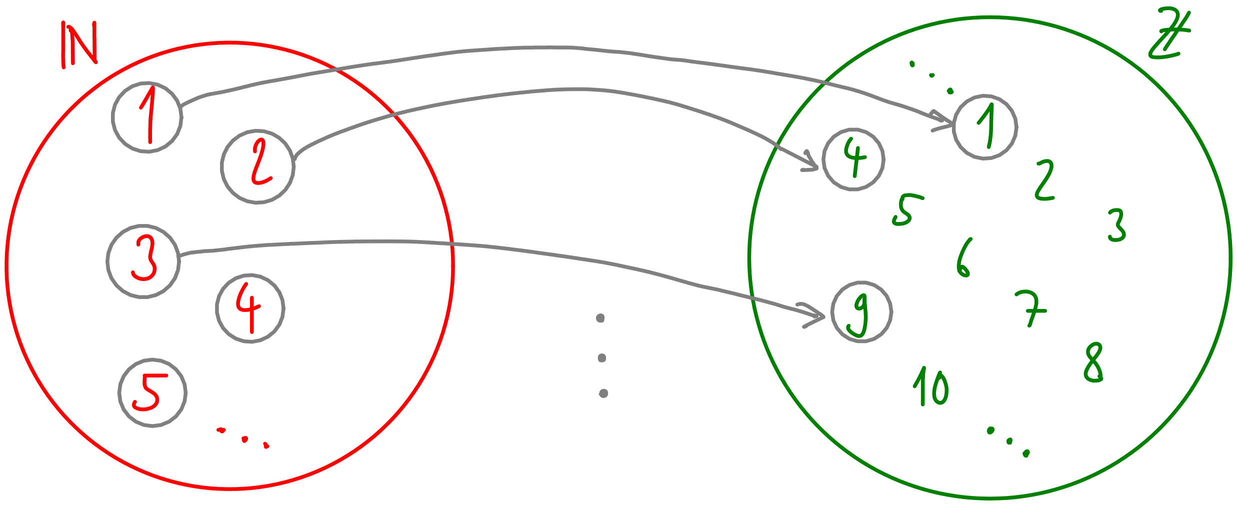
$f(\square) = 6$

Start Learning Sets - Part 5

Map: $f : A \to B$

Example: $f : \mathbb{N} \to \mathbb{Z}$

$x \mapsto x^2$ new notation for $f(x) = x^2$

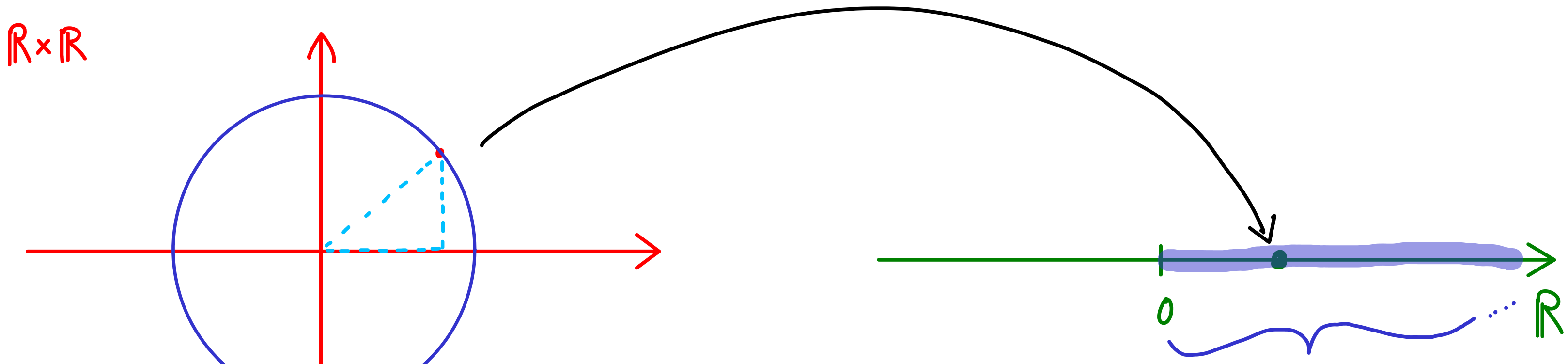


Range: $\mathrm{Ran}(f) := \{ y \in B \mid \exists x \in A : f(x) = y \}$

$=: \{ f(x) \mid x \in A \}$ (shorter notation)

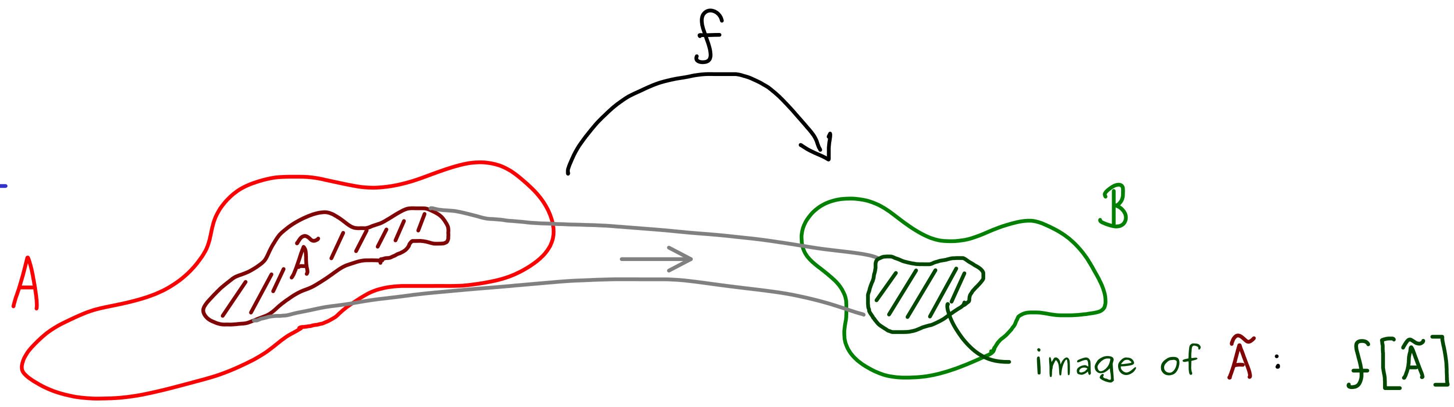
Example: $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$(x_1, x_2) \mapsto x_1^2 + x_2^2$



$\mathrm{Ran}(f) = \{ y \in \mathbb{R} \mid y \geq 0 \}$

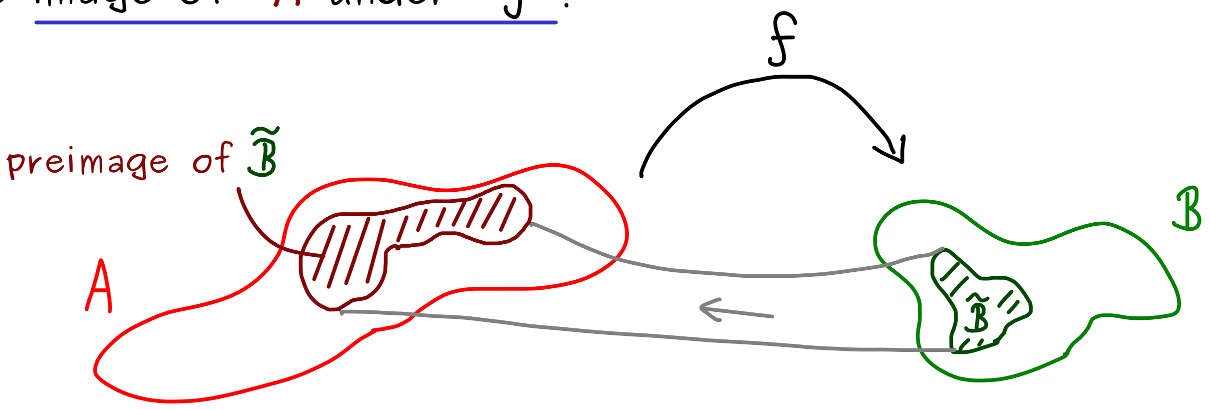
Image and preimage:



For a subset $\tilde{A} \subseteq A$,

$$f[\tilde{A}] := \{ y \in B \mid \exists x \in \tilde{A} : f(x) = y \} = \{ f(x) \mid x \in \tilde{A} \}$$

denotes the image of $\tilde{A}$ under f.



For $\tilde{B} \subseteq B$,

$$f^{-1}[\tilde{B}] := \{ x \in A \mid f(x) \in \tilde{B} \}$$

denotes the preimage of $\tilde{B}$ under f.

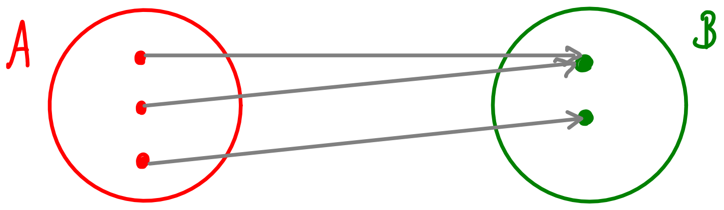
Example: $f : \mathbb{N} \to \mathbb{Z}$

$$x \mapsto \begin{cases} 0 & \text{if } x \text{ even} \\ x & \text{if } x \text{ odd} \end{cases}$$

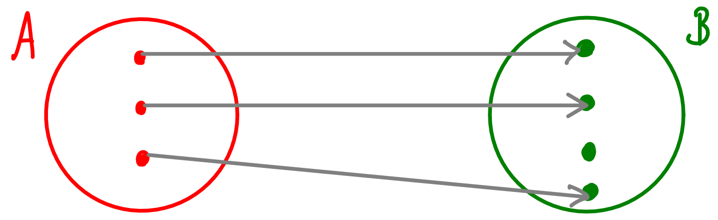
$$f[\{2,3,4\}] = \{0,3\}$$

$$f^{-1}[\{0\}] = \{2,4,6,8,10,\dots\}$$

Start Learning Sets – Part 6



not injective



not surjective

Definition: A map $f: A \to B$ is called:

injective if $\forall x_1, x_2 \in A : \left(x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2) \right)$ is true

surjective if $\forall y \in B : \exists x \in A : f(x) = y$ is true

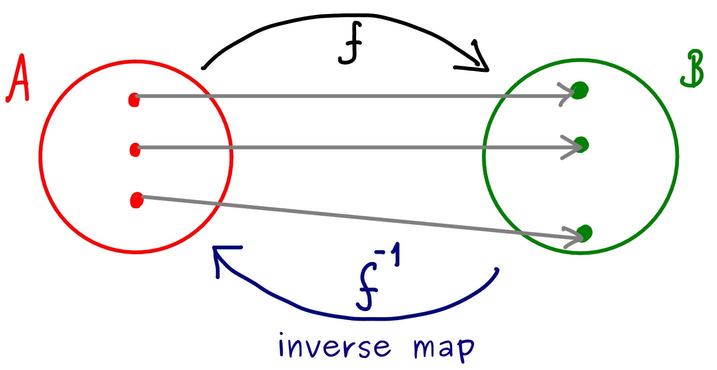
Remember:

surjective: Each $y \in B$ gets at least one arrow.

injective: Each $y \in B$ gets at most one arrow.

$\underbrace{\text{injective + surjective}}_{\text{bijective } (1:1)}$ Each $y \in B$ gets exactly one arrow.

bijective $(1:1)$
$\Vert$
invertible



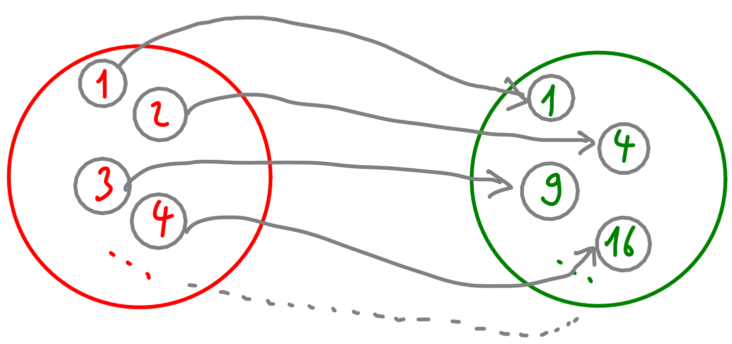
bijective

$$f^{-1}: B \to A ,$$

$$f^{-1}(y) := x \quad \text{if} \quad f(x) = y$$

Example: $f: \mathbb{N} \to \{1, 4, 9, 16, 25, 36, \dots\}$

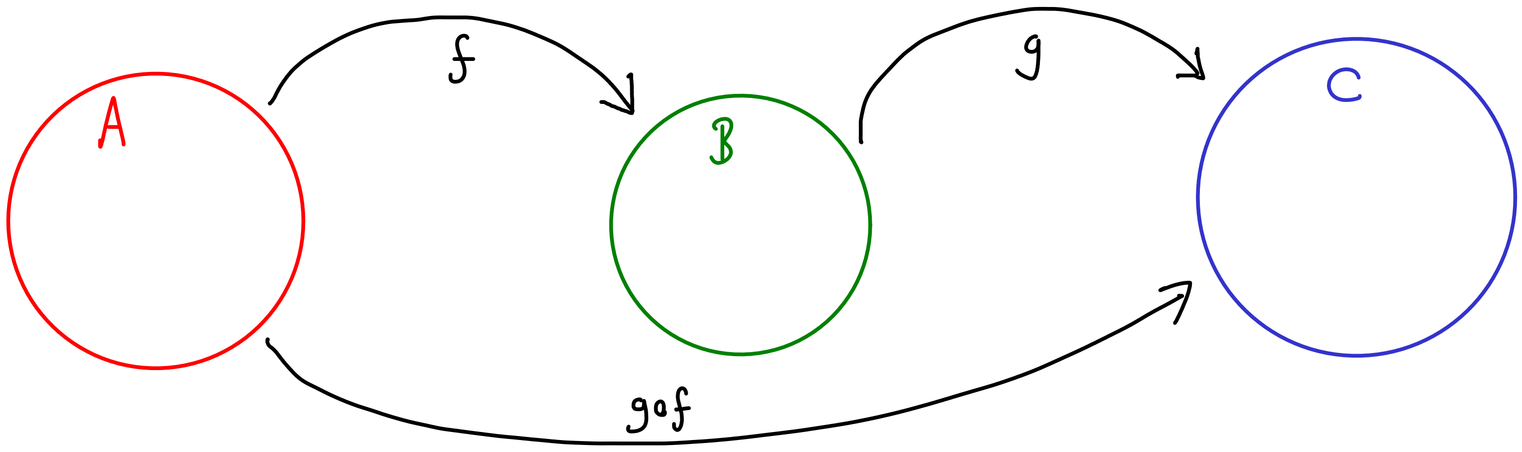
$x \mapsto x^2$



$f^{-1}: \{1, 4, 9, 16, 25, 36, \dots\} \to \mathbb{N}$

$y \mapsto \sqrt{y}$

Start Learning Sets – Part 7

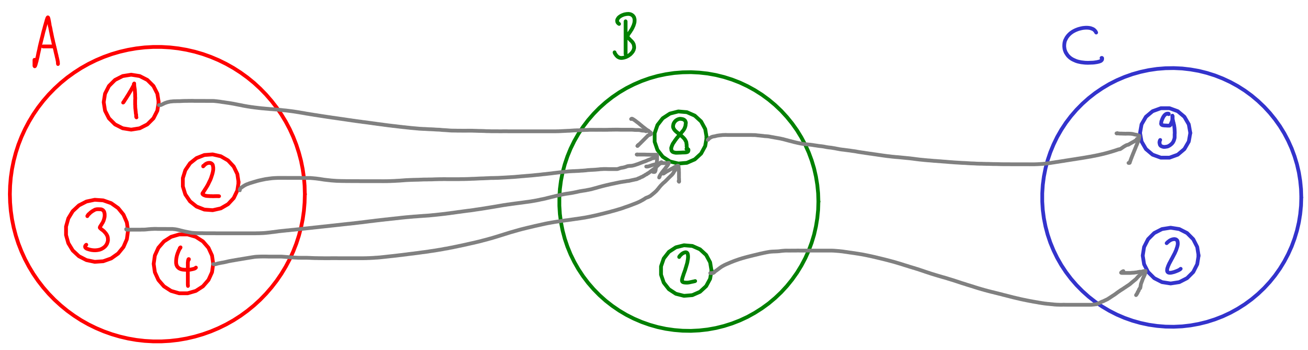


For $f: A \rightarrow B$ and $g: B \rightarrow C$ define:

$$\left.\begin{aligned} g\circ f: \; & A \longrightarrow C \\ & x \longmapsto g(f(x)) \end{aligned}\right\} \text{ called the composition } g \text{ with } f$$

Examples:

(1)



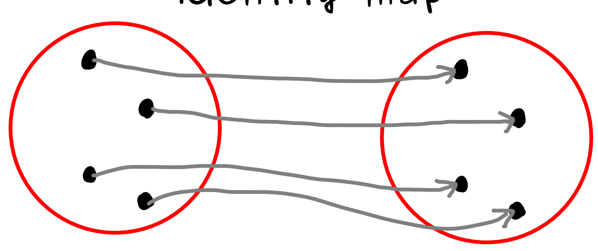
$g(8) = 9$
$g(2) = 2$
$f(1) = 8$
$f(2) = 8$
$f(3) = 8$
$f(4) = 8$

$(g\circ f)(1) = 9$ $\quad (g\circ f)(2) = 9$
$(g\circ f)(3) = 9$ $\quad (g\circ f)(4) = 9$

(2) $f: \mathbb{R} \rightarrow \mathbb{R}, \; x \mapsto x^2$, $\quad g: \mathbb{R} \rightarrow \mathbb{R}, \; x \mapsto \sin(x)$

$\rightsquigarrow \; (g\circ f)(x) = \sin(x^2)$ and $(f\circ g)(x) = (\sin(x))^2$

identity map



For any set A, we define: $\quad id_A: A \rightarrow A, \; x \mapsto x$

For $f: A \rightarrow B$ bijective, we have:

$$f\circ f^{-1} = id_B$$
$$f^{-1}\circ f = id_A$$

The Bright Side of Mathematics

The following pages cover the whole Start Learning Numbers course of the Bright Side of Mathematics. Please note that the creator lives from generous supporters and would be very happy about a donation. See more here: https://tbsom.de/support

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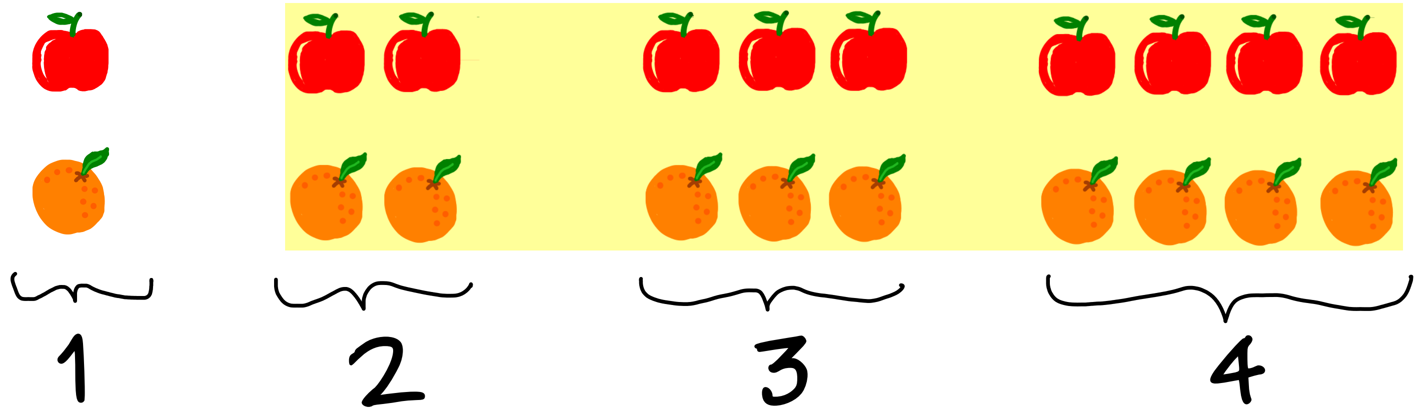
Start Learning Numbers – Part 1

Natural numbers



$$\mathbb{N} = \{1, 2, 3, 4, \dots\}$$

$$\mathbb{N}_0 = \{0, 1, 2, 3, 4, \dots\}$$

$0 := \emptyset$ empty set

$1 := \{0\}$ set with one element

$2 := \{0, 1\}$ set with two elements

$3 := \{0, 1, 2\}$ set with three elements

$4 := \{0, 1, 2, 3\} = 3 \cup \{3\}$

$\vdots$

Axiom: There is a set $\mathbb{N}_0$ with the properties:
(a) $0 \in \mathbb{N}_0$
(b) $\forall x: \; x \in \mathbb{N}_0 \rightarrow x \cup \{x\} \in \mathbb{N}_0$

And $\mathbb{N}_0$ is the smallest set having these two properties.

Successor map: $s: \mathbb{N}_0 \longrightarrow \mathbb{N}_0, \; x \mapsto x \cup \{x\}$, $s(6) = 7$

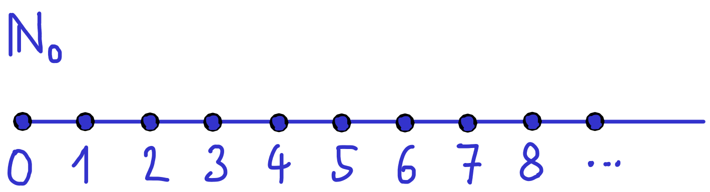
Start Learning Numbers — Part 2

Natural numbers: $\mathbb{N}_0 = \{0, 1, 2, 3, 4, \dots\}$

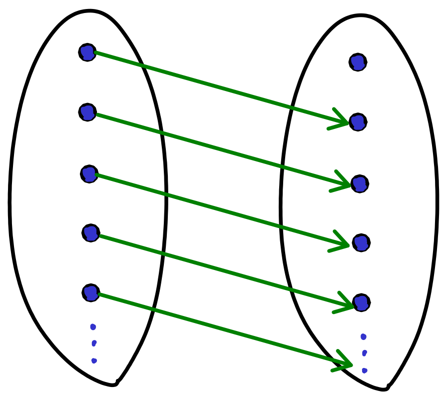
Properties of $\mathbb{N}_0$:

(1) $0 \in \mathbb{N}_0$

(2) There is a map $s: \mathbb{N}_0 \to \mathbb{N}_0$ that satisfies:



(2a) s is injective



(2b) $0 \notin \mathrm{Ran}(s) = s[\mathbb{N}_0]$

(2c) If $M \subseteq \mathbb{N}_0$ with $0 \in M$ and $s[M] \subseteq M$, then $M = \mathbb{N}_0$.

(mathematical induction)

Addition in $\mathbb{N}_0$: map

$$\begin{aligned} \mathbb{N}_0 \times \mathbb{N}_0 &\longrightarrow \mathbb{N}_0 \\ (m, n) &\longmapsto m + n \end{aligned}$$

How is it defined? $2 + 4 := 6$

$\boxed{m + 0 := m}$, $m + 1 := s(m)$, $m + 2 := s(m+1)$

Recursive definition: $\boxed{m + s(n) := s(m+n)}$

$$2 + 5 = 2 + s(4) = s(2+4) = s(6) = 7$$

Dedekind's principle of recursive definition:

For a set A, $a \in A$ and $h: A \to A$, then there exists a unique map $f: \mathbb{N}_0 \to A$ with $f(0) = a$ and $f(s(n)) = h(f(n))$.

$$\left(\text{"}\ a, h(a), h(h(a)), h(h(h(a))), \dots \text{"} \right)$$

Start Learning Numbers – Part 3

Natural numbers: $\mathbb{N}_0 = \{0, 1, 2, 3, 4, \dots\}$

Each $n \in \mathbb{N}_0$ has a unique successor:

$$s : \mathbb{N}_0 \to \mathbb{N}_0 \quad , \qquad s(n) = n + 1$$

We already know: $m + (n+1) = (m+n) + 1$ (RD)

Mathematical induction:

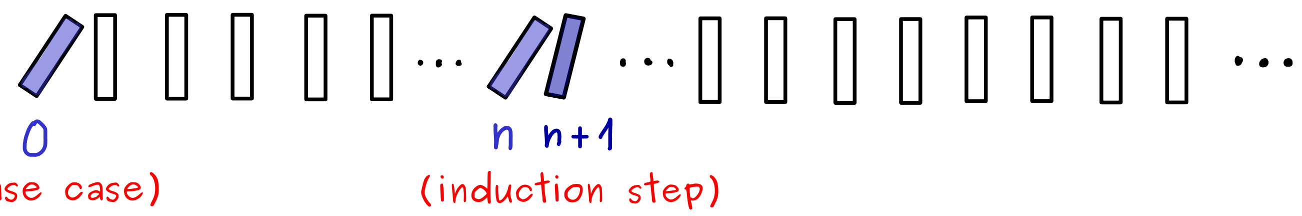
$\mathbb{N}_0$ satisfies the induction property:

Let $P(n)$ be a property for natural numbers n ("predicate").

If: (1) $P(0)$ is true (base case)

(2) $\forall n \in \mathbb{N}_0 : P(n) \rightarrow P(n+1)$ is true (induction step)

Then: $P(n)$ is true for all $n \in \mathbb{N}_0$ ($\forall n : P(n)$ is true)



Proposition: For all $k, m, n \in \mathbb{N}_0$, we have:

$$(k + m) + n = k + (m + n) \qquad \text{(associative law)}$$

Proof: Use mathematical induction.

$P(n)$ is given by:

$$\forall k, m \in \mathbb{N}_0 : \quad (k + m) + n = k + (m + n)$$

Base case: $\mathcal{P}(0)$ means $\forall k,m \in \mathbb{N}_0 : \underbrace{(k+m)+0}_{k+m} = k + (\underbrace{m+0}_{m})$

$\Leftrightarrow \quad \forall k,m \in \mathbb{N}_0 : k+m = k + m$ true

Induction step: $\left(\forall n \in \mathbb{N}_0 : \mathcal{P}(n) \rightarrow \mathcal{P}(n+1)\right)$

Assume $\mathcal{P}(n)$ is true.

$$\boxed{m + (n+1) = (m+n)+1 \quad \text{(RD)}}$$

$\mathcal{P}(n+1)$ means $\forall k,m \in \mathbb{N}_0 : (k+m)+(n+1) = k + (m+(n+1))$

Left-hand side: $(k+m)+(n+1) \overset{\text{(RD)}}{=} \big((k+m)+n\big)+1$

$$\overset{\mathcal{P}(n)}{=} \big(k+(m+n)\big)+1$$

Right-hand side

$$\overset{\text{(RD)}}{=} k+\big((m+n)+1\big) \overset{\text{(RD)}}{=} k+\big(m+(n+1)\big)$$

Start Learning Numbers - Part 4

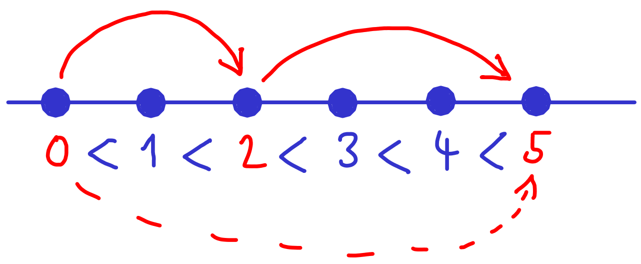
Natural numbers: $\mathbb{N}_0 = \{0, 1, 2, 3, 4, \dots\}$

Addition $+$ is a map $\mathbb{N}_0 \times \mathbb{N}_0 \longrightarrow \mathbb{N}_0$ with:

- $m + 0 = m$ (neutral element)
- $(k + m) + n = k + (m + n)$ (associative law)
- $m + n = n + m$ (commutative law)

Ordering: We write $n \leq m$ if:

$\exists k \in \mathbb{N}_0 : \quad m = n + k$



And we write $n < m$ if: $\quad n \leq m \quad \wedge \quad n \neq m$

Properties:

(1) $n \leq n$ (reflexive)

(2) If $n \leq m \wedge m \leq n$, then $n = m$ (antisymmetric)

(3) If $n \leq \ell \wedge \ell \leq m$, then $n \leq m$ (transitive)

Proof: Assume $n \leq \ell$ and $\ell \leq m$ are true. So:

$\exists k_1 \in \mathbb{N}_0 : \ \ell = n + k_1$ and $\exists k_2 \in \mathbb{N}_0 : \ m = \ell + k_2$ are true.

Therefore: $m = \ell + k_2 = (n + k_1) + k_2$

$$= n + \underbrace{(k_1 + k_2)}_{=: k \in \mathbb{N}_0} = n + k$$

Therefore: $\exists k \in \mathbb{N}_0 : \ m = n + k$ is true, so $n \leq m$ is true.

Start Learning Numbers - Part 5

Natural numbers: $\mathbb{N}_0 = \{0, 1, 2, 3, 4, \dots\}$

$$\underbrace{4+4+4+4+4}_{\text{We have } 5 \text{ of them}} =: 5 \cdot 4$$

$$3+3+3+3+3+3 =: 6 \cdot 3$$

$$4 =: 1 \cdot 4$$

$$0 =: 0 \cdot 4$$

How can we define the multiplication?

Multiplication in $\mathbb{N}_0$: map $\mathbb{N}_0 \times \mathbb{N}_0 \longrightarrow \mathbb{N}_0$

$(n, m) \longmapsto n \cdot m$ defined by

$$0 \cdot m := 0$$
$$(n+1) \cdot m := (n \cdot m) + m$$

(recursive definition)

$$5 \cdot 2 = 2 + 2 + 2 + 2 + 2$$

$$6 \cdot 2 = \underbrace{2 + 2 + 2 + 2 + 2}_{5 \cdot 2} + 2$$

(Map is well-defined by Dedekind's recursion theorem)

Properties:

(1) $n \cdot (m \cdot k) = (n \cdot m) \cdot k$ (associative)

(2) $n \cdot m = m \cdot n$ (commutative)

(3) $1 \cdot m = m$ (neutral element)

How to connect $+$ and $\cdot$: $\quad n\cdot(m+k) = n\cdot m + n\cdot k$ (distributive)

$$\boxed{\begin{aligned} 0\cdot m &:= 0 \\ (n+1)\cdot m &:= (n\cdot m) + m \end{aligned}} \quad (*)$$

Proof by induction: Base case: $n=0$

Left-hand side: $0\cdot(m+k) = 0$ ✓

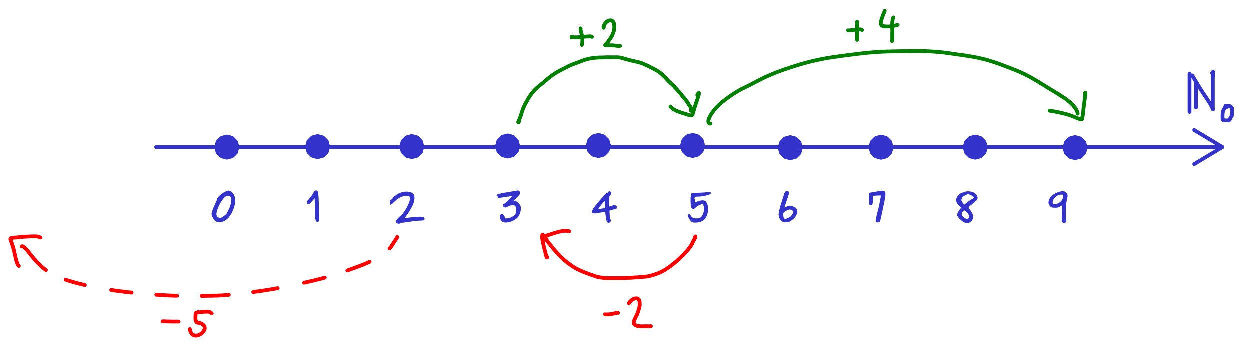
Right-hand side: $0\cdot m + 0\cdot k = 0 + 0 = 0$ ✓

Induction step: Assume $n\cdot(m+k) = n\cdot m + n\cdot k$ holds for n.

(induction hypothesis)

Left-hand side:
$$\begin{aligned}(n+1)\cdot(m+k) &\overset{(*)}{=} n\cdot(m+k) + (m+k) \\ &\overset{(i.h.)}{=} n\cdot m + (n\cdot k + m) + k \\ &= (n\cdot m + m) + (n\cdot k + k) \\ &\overset{(*)}{=} (n+1)\cdot m + (n+1)\cdot k \quad \leftarrow \text{Right-hand side}\end{aligned}$$

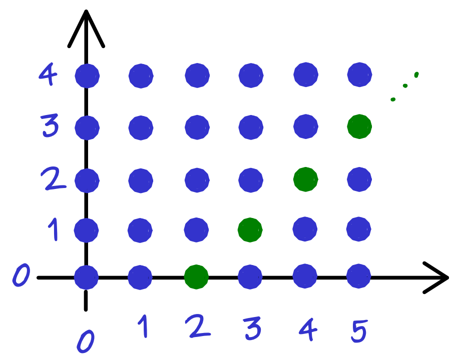
Start Learning Numbers — Part 6



$5 - 2 = 3$ $\qquad$ $2 - 5 = ?$

Idea: Look at pairs $(9,5)$, $(5,3)$, $(3,5)$

$$\mathbb{N}_0 \times \mathbb{N}_0 =: \mathbb{N}_0^2$$



$\vdots$
$(5,3)$
$(4,2)$
$(3,1)$
$(2,0)$
should be the "same"

$(5,3)$ stands for "$5-3$"
$(4,2)$ stands for "$4-2$"

$5 - 3 = 4 - 2$ ← not okay

$5 + 2 = 4 + 3$ ← totally okay

$(5,3) \sim (4,2)$ (equivalent)

Equivalence relation: We write $(a,b) \sim (x,y)$ if:

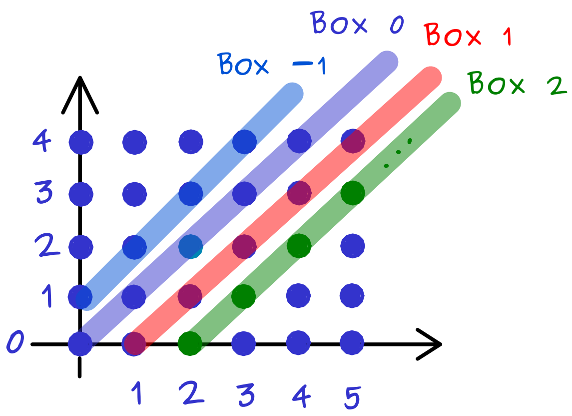
$$a + y = x + b$$

Properties:

(1) $(a,b) \sim (a,b)$ (reflexive)

(2) If $(a,b) \sim (x,y)$, then $(x,y) \sim (a,b)$. (symmetric)

(3) If $(a,b) \sim (x,y)$ and $(x,y) \sim (c,d)$, then $(a,b) \sim (c,d)$. (transitive)



Property of $\mathbb{N}_0$ (cancellation):

If $m + n = \tilde{m} + n$, then $m = \tilde{m}$.

$$\text{Box } 0 = [(2,2)]_\sim := \{ (x,y) \in \mathbb{N}_0^2 \mid (x,y) \sim (2,2) \}$$

is called the equivalence class of $(2,2)$.

$$\text{Box } 0 = [(0,0)]_\sim = [(2,2)]_\sim$$

$$\text{Box } 1 = [(1,0)]_\sim = [(9,8)]_\sim$$

$$\text{Box } 2 = [(2,0)]_\sim$$

$\vdots \qquad \vdots$

$$\text{Box } -1 = [(0,1)]_\sim = [(8,9)]_\sim$$

$$\text{Box } -2 = [(0,2)]_\sim$$

$\vdots \qquad \vdots$

$\mathbb{Z} :=$ set of all boxes (equivalence classes)

Start Learning Numbers – Part 7

In $\mathbb{N}_0$ $\quad 4 + x = 0$ is not solvable! No "inverse" of 4.

$$\mathbb{Z} := \left\{ [(a,b)]_\sim \;\middle|\; (a,b) \in \mathbb{N}_0^2 \right\} =: \mathbb{N}_0^2 / \sim$$

with $[(a,b)]_\sim := \left\{ (x,y) \;\middle|\; (x,y) \sim (a,b) \right\}$

and $(x,y) \sim (a,b) \;\Leftrightarrow\; x + b = a + y$

$[(0,0)]_\sim =: 0_\mathbb{Z}$ $\qquad$ $[(0,1)]_\sim =: (-1)_\mathbb{Z}$

$[(1,0)]_\sim =: 1_\mathbb{Z}$ $\qquad$ $[(0,2)]_\sim =: (-2)_\mathbb{Z}$

$[(2,0)]_\sim =: 2_\mathbb{Z}$ $\qquad$ $\vdots$

$\vdots$

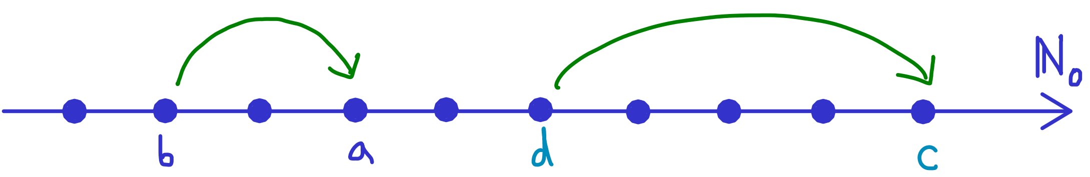
$$\mathbb{Z} = \left\{ \dots, (-2)_\mathbb{Z}, (-1)_\mathbb{Z}, 0_\mathbb{Z}, 1_\mathbb{Z}, 2_\mathbb{Z}, \dots \right\}$$

Question: Is $4_\mathbb{Z} + x = 0_\mathbb{Z}$ now solvable? And with $x = (-4)_\mathbb{Z}$?

First question: How is $+$ as a map $\mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}$ defined?

$$[(a,b)]_\sim + [(c,d)]_\sim := [(a+c, b+d)]_\sim$$

well-defined? ✓



Take $(\tilde{a},\tilde{b}) \sim (a,b)$ and $(\tilde{c},\tilde{d}) \sim (c,d)$. Then $[(\tilde{a},\tilde{b})]_\sim + [(\tilde{c},\tilde{d})]_\sim = [(\tilde{a}+\tilde{c}, \tilde{b}+\tilde{d})]_\sim$

Is $(\tilde{a}+\tilde{c}, \tilde{b}+\tilde{d}) \sim (a+c, b+d)$?

Proof: $(\tilde{a},\tilde{b}) \sim (a,b) \iff \tilde{a}+b = a+\tilde{b}$

$(\tilde{c},\tilde{d}) \sim (c,d) \iff \tilde{c}+d = c+\tilde{d}$

implies: $\tilde{a}+\tilde{c}+b+d = a+c+\tilde{b}+\tilde{d}$

$\iff (\tilde{a}+\tilde{c}, \tilde{b}+\tilde{d}) \sim (a+c, b+d)$

Examples: (a) $4_{\mathbb{Z}} + 2_{\mathbb{Z}} = [(4,0)]_\sim + [(2,0)]_\sim = [(6,0)]_\sim = 6_{\mathbb{Z}}$

(b) $4_{\mathbb{Z}} + (-4)_{\mathbb{Z}} = [(4,0)]_\sim + [(0,4)]_\sim = [(4,4)]_\sim = [(0,0)]_\sim = 0_{\mathbb{Z}}$

Properties of $\mathbb{Z}$ together with $+$: map $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$

(a) associative

(b) commutative

(c) $m + 0_{\mathbb{Z}} = m$ ($0_{\mathbb{Z}}$ is neutral element)

(d) For all $m \in \mathbb{Z}$, there is an element $\tilde{m} \in \mathbb{Z}$ with $m + \tilde{m} = 0_{\mathbb{Z}}$

$\to$ $(\mathbb{Z}, +)$ is an abelian group

Start Learning Numbers - Part 8

$$\mathbb{Z} = \{ \dots, (-2)_\mathbb{Z}, (-1)_\mathbb{Z}, 0_\mathbb{Z}, 1_\mathbb{Z}, 2_\mathbb{Z}, \dots \}$$

$2_\mathbb{Z} = [(6,4)]_\sim$ think of "6-4"

think of "$(a-b)\cdot(c-d) = (ac+bd) - (ad+bc)$"

$$[(a,b)]_\sim \cdot [(c,d)]_\sim := [(a\cdot c + b\cdot d \,,\, a\cdot d + b\cdot c)]_\sim$$

The multiplication is well-defined.

Properties of $\mathbb{Z}$ together with $\cdot$:

map $\mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$

(a) associative

(b) commutative

(c) $1_\mathbb{Z}\cdot m = m$ ($1_\mathbb{Z}$ is neutral element)

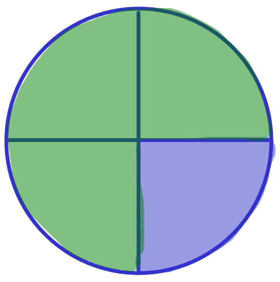
(d) distributive

Examples: (a) $4_\mathbb{Z} \cdot 2_\mathbb{Z} = [(4,0)]_\sim \cdot [(2,0)]_\sim = [(4\cdot 2 + 0\cdot 0 \,,\, 4\cdot 0 + 0\cdot 2)]_\sim = 8_\mathbb{Z}$

(b) $(-4)_\mathbb{Z}\cdot(-2)_\mathbb{Z} = [(0,4)]_\sim \cdot [(0,2)]_\sim = [(0\cdot 0 + 4\cdot 2 \,,\, 0\cdot 2 + 4\cdot 0)]_\sim = 8_\mathbb{Z}$

Start Learning Numbers – Part 9

$$\mathbb{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$$



ratio: $3:1$ or $3:4$ or $1:4$

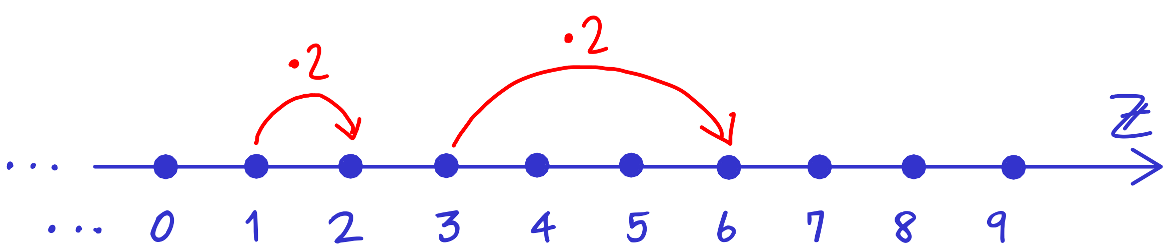
fraction: $\frac{3}{4} + \frac{1}{4} = 1$

Solve $4 \cdot x = 1$? $\rightsquigarrow$ We need inverses with respect to $\cdot$!

Works the same as $(\mathbb{N}_0, +) \rightsquigarrow (\mathbb{Z}, +)$

For $(c, d), (a, b) \in \mathbb{Z} \times \mathbb{Z}\setminus\{0\}$ define:

$$(a,b) \sim (c,d) \quad \text{by} \quad a \cdot d = c \cdot b$$



$(6,3) \sim (2,1)$ "$\frac{6}{3} = \frac{2}{1}$"

$$\mathbb{Q} := (\mathbb{Z} \times \mathbb{Z}\setminus\{0\})/\sim \; = \left\{ [(a,b)]_\sim \;\middle|\; (a,b) \in \mathbb{Z} \times \mathbb{Z}\setminus\{0\} \right\}$$ rational numbers

Examples: $[(4,2)]_\sim = [(6,3)]_\sim = [(2,1)]_\sim =: 2_\mathbb{Q}$

$[(-9,-3)]_\sim = [(9,3)]_\sim = [(3,1)]_\sim =: 3_\mathbb{Q}$, $[(0,8)]_\sim = [(0,1)]_\sim =: 0_\mathbb{Q}$

$[(-9,3)]_\sim = [(-3,1)]_\sim =: (-3)_\mathbb{Q}$ We get all integers back!

$[(2,8)]_\sim = [(1,4)]_\sim =: \left(\frac{1}{4}\right)_\mathbb{Q}$ $\rightsquigarrow$ fractions

Definition: $[(a,b)]_\sim =: \frac{a}{b}$ $\left(\frac{2}{8} = \frac{1}{4} \right)$

Start Learning Numbers – Part 10

$$\mathbb{Q} = \left\{ \frac{a}{b} \;\middle|\; a \in \mathbb{Z},\ b \in \mathbb{Z}\setminus\{0\} \right\}, \qquad \frac{a}{b} = \frac{c}{d} \iff a\cdot d = c\cdot b$$

Multiplication: $\frac{a}{b} \cdot \frac{c}{d} := \frac{a\cdot c}{b\cdot d}$ well-defined!

For $a \neq 0$, we have: $\frac{a}{b}\cdot\frac{b}{a} = \frac{a\cdot b}{b\cdot a} = \frac{1}{1} \left(= 1_{\mathbb{Q}} \right)$

Solve: $4\cdot x = 1$? In $\mathbb{Q}$: $\frac{4}{1}\cdot x = \frac{1}{1}$ is solved by: $x = \frac{1}{4}$

Property: $(\mathbb{Q}\setminus\{0_{\mathbb{Q}}\}, \cdot)$ is an abelian group.

How to define the addition?

We want the distributive law:

$$\boxed{\frac{a}{d} + \frac{c}{d}} = \frac{a}{1}\cdot\frac{1}{d} + \frac{c}{1}\cdot\frac{1}{d} = \left(\frac{a}{1} + \frac{c}{1}\right)\cdot\frac{1}{d} = \boxed{\frac{a+c}{d}}$$

should be defined by:

$$\frac{a}{b} + \frac{c}{d} = \frac{a}{b}\cdot\frac{d}{d} + \frac{c}{d}\cdot\frac{b}{b} = \frac{a\cdot d}{1}\cdot\frac{1}{b\cdot d} + \frac{c\cdot b}{1}\cdot\frac{1}{b\cdot d}$$
$$= \left(\frac{a\cdot d}{1} + \frac{c\cdot b}{1}\right)\cdot\frac{1}{b\cdot d} = \frac{a\cdot d + c\cdot b}{b\cdot d}$$

Define: $\frac{a}{b} + \frac{c}{d} := \frac{a\cdot d + c\cdot b}{b\cdot d}$ well-defined!

Proposition: The set $\mathbb{Q}$ together with the operation $+$ and $\cdot$ satifies:

(1) $(\mathbb{Q}, +)$ is an abelian group

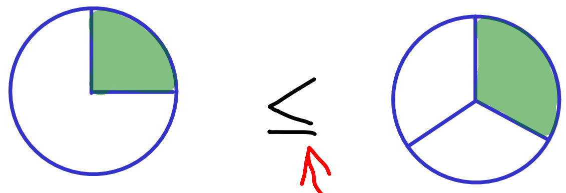
(2) $(\mathbb{Q}\setminus\{0_{\mathbb{Q}}\}, \cdot)$ is an abelian group

(3) distributive law

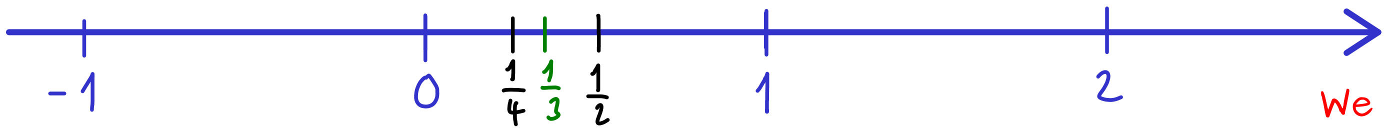
field

Start Learning Numbers – Part 11

$$\mathbb{Q} = \left\{ \frac{a}{b} \;\middle|\; a \in \mathbb{Z},\ b \in \mathbb{Z} \setminus \{0\} \right\}$$



We need to define it



Definition of $\leq$ for $\mathbb{Z}$: For $a, b \in \mathbb{Z}$, we write $a \leq b$ if $\exists k \in \mathbb{N}_0 : a + k = b$

Now: $\frac{1}{4} \leq \frac{1}{3}$ because $3 \leq 4$

Definition of $\leq$ for $\mathbb{Q}$: For $b > 0$ and $d > 0$

$$\frac{a}{b} \leq \frac{c}{d} \quad \text{defined by} \quad a \cdot d \leq c \cdot b$$

Properties of $\leq$ for $\mathbb{Q}$:

(1) Ordering: reflexive, antisymmetric and transitive.

(2) For all $x, y, z \in \mathbb{Q}$: If $x \leq y$, then $x + z \leq y + z$

(3) For all $x, y, z \in \mathbb{Q}$: If $z \geq 0$ and $x \leq y$, then $x \cdot z \leq y \cdot z$

(4) Total order: For all $x, y \in \mathbb{Q}$, we have $x \leq y$ or $y \leq x$.

(5) Archimedean property: For all $x, \varepsilon \in \mathbb{Q}$ with $x > 0$ and $\varepsilon > 0$,

we have: $n \in \mathbb{N}_0 : n \cdot \varepsilon = \varepsilon + \varepsilon + \varepsilon + \cdots + \varepsilon > x$

The Bright Side of Mathematics

The following pages cover the whole Start Learning Reals course of the Bright Side of Mathematics. Please note that the creator lives from generous supporters and would be very happy about a donation. See more here: https://tbsom.de/support

Have fun learning mathematics!

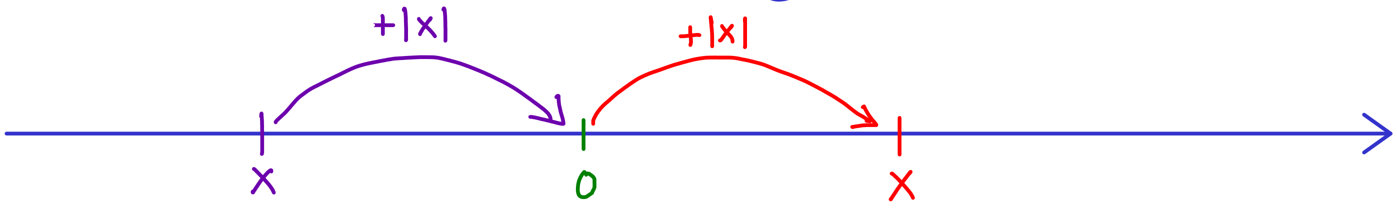
Start Learning Reals - Part 1

Real numbers $\mathbb{R}$

Starting point: $\mathbb{Q}$ is the set of fractions $\rightsquigarrow$ field and Archimedean order $\leq$

$x > 0 \quad , \quad x < 0$

Absolute value: For $x \in \mathbb{Q}$ define: $|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$



How far away is x from 0 ? $\rightsquigarrow |x|$

Problem: There is no $x \in \mathbb{Q}$ with $x^2 = 2$

$$x_1 = \frac{14}{10} = \frac{7}{5} \quad \rightsquigarrow \quad x_1^2 = \frac{49}{25} \approx 2$$

$$x_2 = \frac{141}{100} \quad \rightsquigarrow \quad x_2^2 = \frac{19\,881}{10\,000} \approx 2$$

$$x_3 = \frac{1414}{1000} \quad \rightsquigarrow \quad x_3^2 = \frac{499\,849}{250\,000} \approx 2$$

distance:

$|x_5 - x_3|$

$$x_4 = \frac{14142}{10000} \quad \rightsquigarrow \quad x_4^2 = \frac{49\,999\,041}{25\,000\,000} \approx 2$$

$$x_5 = \frac{141\,421}{100\,000} \quad \rightsquigarrow \quad x_5^2 = \frac{19\,999\,899\,241}{10\,000\,000\,000} \approx 2$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

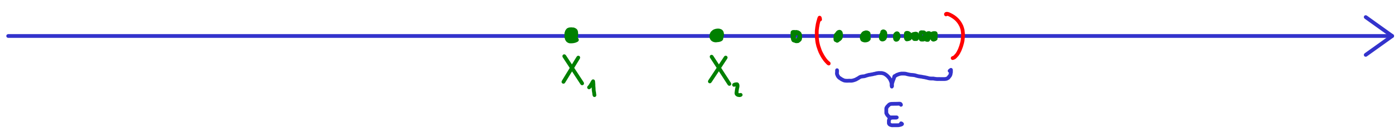
$$x = \, ? \quad \rightsquigarrow \quad x^2 = 2$$

We consider a sequence $(x_n)_{n\in\mathbb{N}}$ (infinite list; formally: a map $\mathbb{N} \to \mathbb{Q}$, $n \mapsto x_n$)

with the property:

$$\forall \varepsilon \in \mathbb{Q} \quad \exists N \in \mathbb{N} \quad \forall n,m \in \mathbb{N} : \left(\varepsilon > 0 \ \wedge \ n,m \geq N \implies |x_n - x_m| < \varepsilon \right)$$

In short: $\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n,m \geq N : \quad |x_n - x_m| < \varepsilon \qquad (*)$



<u>Cauchy sequence:</u> sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \in \mathbb{Q}$ and property $(*)$

Start Learning Reals - Part 2

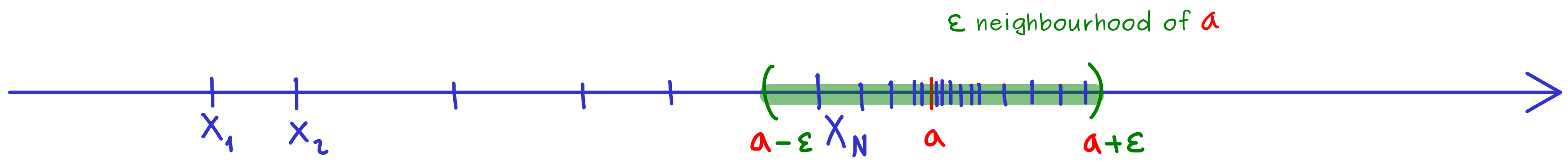
Absolute value in $\mathbb{Q}$: $\quad |x \cdot y| = |x| \cdot |y| \quad$ (multiplicative)

$|x+y| \leq |x| + |y| \quad$ (triangle inequality)

Cauchy sequence: $(x_n)_{n \in \mathbb{N}}$ with $\forall \varepsilon > 0 \;\; \exists N \in \mathbb{N} \;\; \forall n, m \geq N : \;\; |x_n - x_m| < \varepsilon$

Convergent sequence: $(x_n)_{n \in \mathbb{N}}$ with $\exists a \in \mathbb{Q} \;\; \forall \varepsilon > 0 \;\; \exists N \in \mathbb{N} \;\; \forall n \geq N : \;\; |x_n - a| < \varepsilon$

a is called the limit of $(x_n)_{n \in \mathbb{N}}$



Example: $\left(\frac{1}{n}\right)_{n \in \mathbb{N}}$ is a convergent sequence with limit $a = 0$.

Important fact: Cauchy sequence $\Leftarrow$ Convergent sequence

not correct $\mathbb{Q}$ but in $\mathbb{R}$

Proof for $\Leftarrow$: $\quad |x_n - x_m| = |x_n - a + a - x_m| \overset{\text{triangle inequality}}{\leq} |x_n - a| + |a - x_m|$

Let $(x_n)_{n \in \mathbb{N}}$ be a convergent sequence with limit a.

Let $\varepsilon > 0$. $\quad$ Set $\varepsilon' := \frac{\varepsilon}{2} > 0$.

Since $(x_n)_{n \in \mathbb{N}}$ is convergent, there is $N \in \mathbb{N}$ such that:

$$\forall n \geq N : \; |x_n - a| < \varepsilon'$$

Therefore for all $n, m \geq N$:

$$|x_n - x_m| \leq \underbrace{|x_n - a|}_{< \varepsilon'} + \underbrace{|a - x_m|}_{< \varepsilon'} < 2 \cdot \varepsilon' = \varepsilon \quad \Rightarrow (x_n)_{n \in \mathbb{N}} \text{ Cauchy sequence}$$

Axiomatic solution: A non-empty set $\mathbb{R}$ together with operations $+, \cdot$ and ordering $\leq$ is called the real numbers if it satisfies:

(A) $(\mathbb{R}, +, 0)$ is an abelian group

(M) $(\mathbb{R}\setminus\{0\}, \cdot, 1)$ is an abelian group $(1 \neq 0)$

(D) distributive law $x \cdot (y + z) = x \cdot y + x \cdot z$

(O) $\leq$ is a total order, compatible with $+$ and $\cdot$, Archimedean property

(C) Every Cauchy sequence is a convergent sequence. $|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

The complete, whole, full number line $\mathbb{R}$

Start Learning Reals - Part 3

complete number line $\mathbb{R}$

Axioms of the reals: A non-empty set $\mathbb{R}$ together with operations $+$, $\cdot$ and ordering $\leq$ is called the real numbers if it satisfies:

(A) $(\mathbb{R}, +, 0)$ is an abelian group

(M) $(\mathbb{R}\setminus\{0\}, \cdot, 1)$ is an abelian group $(1 \neq 0)$

(D) distributive law $\quad x\cdot(y+z) = x\cdot y + x\cdot z$

(O) $\leq$ is a total order, compatible with $+$ and $\cdot$, Archimedean property

(C) Every Cauchy sequence is a convergent sequence. $\quad |x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

Important facts: There is a set with all these properties (Existence) (Construction) see next video

and it is uniquely determined by these properties. (Uniqueness) (Identification/ Isomorphism)

Show: For all $x \in \mathbb{R}$, we have: $0 \cdot x = 0$ (*) (by only using the axioms).

Proof:

$$0 \underset{\text{inverse}}{\overset{(A)}{=}} (0\cdot x) + (-0\cdot x) \underset{\text{neutral}}{\overset{(A)}{=}} \big((0+0)\cdot x\big) + (-0\cdot x)$$

$$\overset{(D)}{=} (0\cdot x + 0\cdot x) + (-0\cdot x)$$

$$\underset{\text{associativity}}{\overset{(A)}{=}} 0\cdot x + \big(0\cdot x + (-0\cdot x)\big) \underset{\text{inverse}}{\overset{(A)}{=}} 0\cdot x + 0 \underset{\text{neutral}}{\overset{(A)}{=}} 0\cdot x$$

Show: For all $x \in \mathbb{R}$, we have: $(-1)\cdot x = -x$ (by only using the axioms).

Proof:

$$-x \underset{\text{neutral}}{\overset{(A)}{=}} 0 + (-x) \overset{(*)}{=} 0\cdot x + (-x) \underset{\text{inverse}}{\overset{(A)}{=}} \big((-1)+1\big)\cdot x + (-x)$$

$$\overset{(D)}{=} (-1)\cdot x + 1\cdot x + (-x) \overset{(A),(M)}{=} (-1)\cdot x + 0 \underset{\text{neutral}}{\overset{(A)}{=}} (-1)\cdot x$$

Start Learning Reals - Part 4

Construction: $\mathbb{Q} \rightsquigarrow \mathbb{R}$ (Make every Cauchy sequence convergent)

Sequence: $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \dots\right)$ $\rightsquigarrow$ Cauchy sequence and convergent with limit $\frac{1}{3}$

Sequence: $\left(\underbrace{0.3}_{\frac{3}{10}}, \underbrace{0.33}_{\frac{33}{100}}, \underbrace{0.333}_{\frac{333}{1000}}, \dots\right)$ $\rightsquigarrow$ Cauchy sequence and convergent with limit $\frac{1}{3}$

$$\mathcal{C} := \left\{ (x_n)_{n\in\mathbb{N}} \;\middle|\; \forall n \in \mathbb{N} : x_n \in \mathbb{Q} \text{ and } (x_n)_{n\in\mathbb{N}} \text{ is a Cauchy sequence} \right\}$$

For two elements $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$, define:

$$(a_n)_{n\in\mathbb{N}} \sim (b_n)_{n\in\mathbb{N}} \;:\Longleftrightarrow\; (a_n - b_n)_{n\in\mathbb{N}} \text{ convergent with limit } 0$$

$\Longrightarrow$ $\sim$ is an equivalence relation on $\mathcal{C}$ (reflexive, symmetric, transitive)

$\Longrightarrow$ equivalence class $\left[(x_n)_{n\in\mathbb{N}}\right]_\sim := \left\{ (a_n)_{n\in\mathbb{N}} \;\middle|\; (a_n)_{n\in\mathbb{N}} \sim (x_n)_{n\in\mathbb{N}} \right\}$

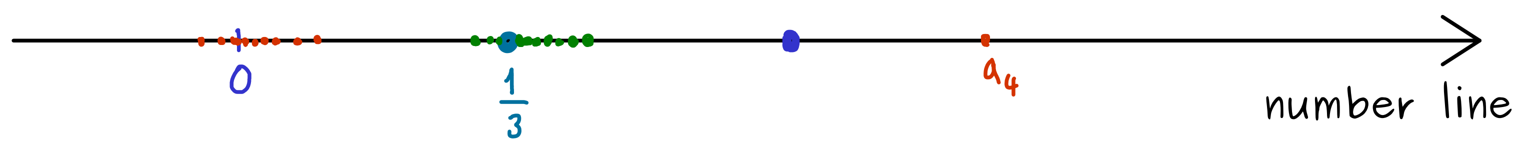
Definition:

$$\mathbb{R} := \mathcal{C}/_\sim := \left\{ \left[(x_n)_{n\in\mathbb{N}}\right]_\sim \;\middle|\; (x_n)_{n\in\mathbb{N}} \in \mathcal{C} \right\}$$

$$\left[(a_n)_{n\in\mathbb{N}}\right]_\sim + \left[(b_n)_{n\in\mathbb{N}}\right]_\sim := \left[(a_n + b_n)_{n\in\mathbb{N}}\right]_\sim \quad \text{(well-defined)}$$

$$\left[(a_n)_{n\in\mathbb{N}}\right]_\sim \cdot \left[(b_n)_{n\in\mathbb{N}}\right]_\sim := \left[(a_n \cdot b_n)_{n\in\mathbb{N}}\right]_\sim \quad \text{(well-defined)}$$

$$\left[(a_n)_{n\in\mathbb{N}}\right]_\sim < \left[(b_n)_{n\in\mathbb{N}}\right]_\sim \;:\Longleftrightarrow\; \exists \delta > 0 \;\exists N \in \mathbb{N} \;\forall n \geq N : \delta < b_n - a_n$$



Properties:

(A) $(\mathbb{R}, +, 0)$ is an abelian group

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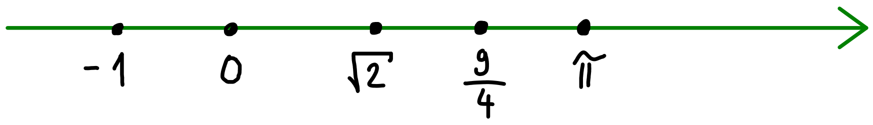
The Bright Side of Mathematics

The following pages cover the whole Start Learning Complex Numbers course of the Bright Side of Mathematics. Please note that the creator lives from generous supporters and would be very happy about a donation. See more here: https://tbsom.de/support

Have fun learning mathematics!

Start Learning Complex Numbers – Part 1

Number line $\mathbb{R}$

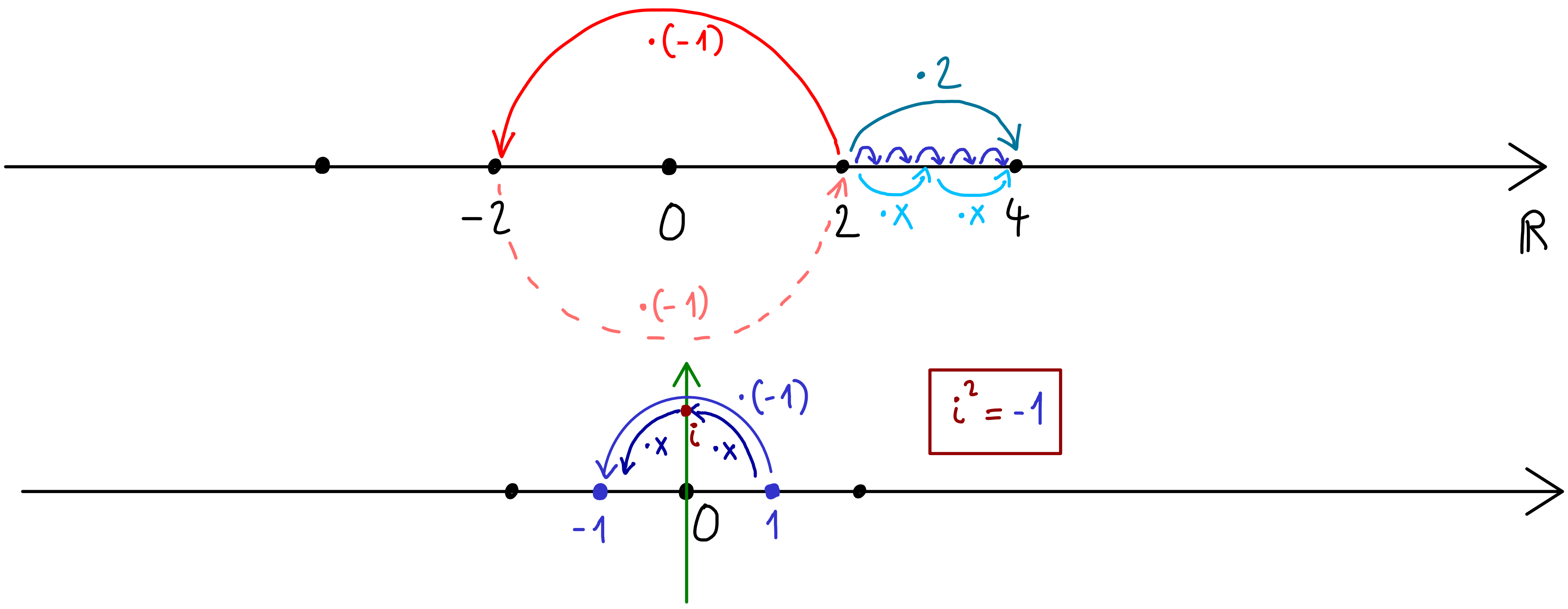


- field $+$ $\cdot$
- ordering $\leq$ (Archimedean, compatible with $+$ and $\cdot$,...)
- complete

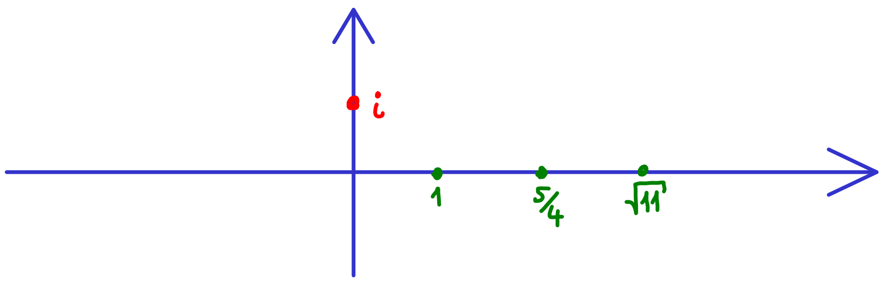
One can solve a lot of equations:

$$x + 5 = 1 \;, \quad x + x = -1$$
$$x \cdot 5 = 1$$
$$x^2 = 2$$

We cannot solve: $\underbrace{x^2}_{x \cdot x} = -1$ (because $x^2 \geq 0$ for all $x \in \mathbb{R}$ and $-1 < 0$)



Start Learning Complex Numbers – Part 2

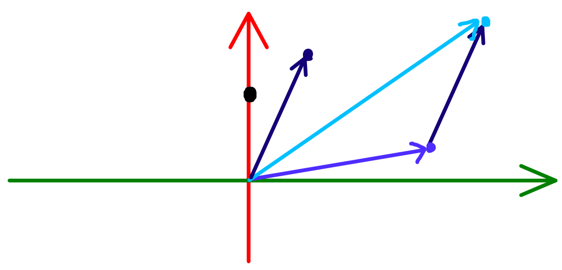


$\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$

$+$ addition

$\cdot$ multiplication

Addition: For $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R} \times \mathbb{R}$, we set: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} := \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$



Multiplication: For $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R} \times \mathbb{R}$, we set: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} := \begin{pmatrix} x_1 \cdot y_1 - x_2 \cdot y_2 \\ x_2 \cdot y_1 + x_1 \cdot y_2 \end{pmatrix}$

Why?

Short notation: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} =: x_1 + i \cdot x_2$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 + i \cdot 1 =: i$

Calculation:

$$(x_1 + i \cdot x_2) \cdot (y_1 + i \cdot y_2) \overset{\text{we want distributivity}}{=} x_1 \cdot y_1 + i \cdot x_2 \cdot y_1 + i \cdot x_1 \cdot y_2 + \underbrace{i^2}_{\substack{\text{we want} \\ = -1}} \cdot x_2 \cdot y_2$$

$$= (x_1 \cdot y_1 - x_2 \cdot y_2) + i \cdot (x_2 \cdot y_1 + x_1 \cdot y_2)$$

Check: $i^2 = i \cdot i = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 - 1 \\ 0 + 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} = -1 + i \cdot 0 = -1$

Properties:

- We write $\mathbb{C} := \mathbb{R}^2$ when we have $+$ and $\cdot$ from above.

field $\left\{ \right.$

- $(\mathbb{C}, +, \underbrace{0}_{= 0 + i \cdot 0})$ is an abelian group (commutative, associative, neutral element, inverses)
- $(\mathbb{C} \setminus \{0\}, \cdot, \underbrace{1}_{= 1 + i \cdot 0})$ is an abelian group (commutative, associative, neutral element, inverses)
- distributive law

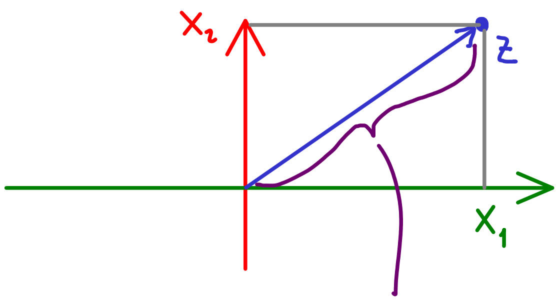
- no nice ordering $\leq$ like for $\mathbb{R}$

Start Learning Complex Numbers – Part 3

$$z = x_1 + i \cdot x_2 \in \mathbb{C}$$

real part of z: $\mathrm{Re}(z)$

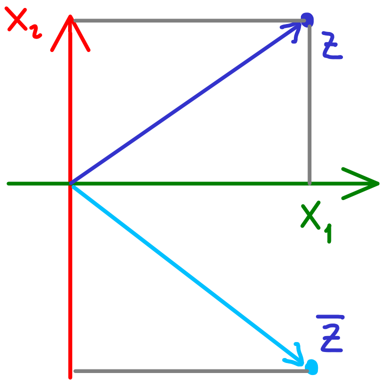
imaginary part of z: $\mathrm{Im}(z)$



length, absolute value, modulus

$$|z| := \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2} \in \mathbb{R}$$

Reflection:
complex conjugate



$$z = x_1 + i \cdot x_2$$

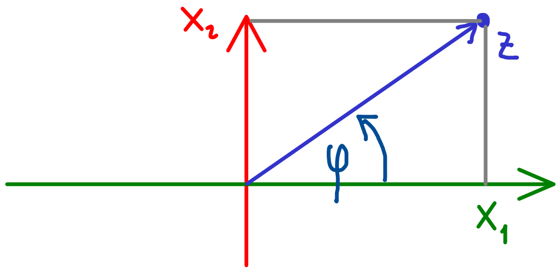
$$\bar{z} = x_1 + i \cdot (-x_2) = x_1 - i \cdot x_2$$

Calculate: $z \cdot \bar{z} = (x_1 + i \cdot x_2) \cdot (x_1 - i \cdot x_2)$

$$= x_1^2 + x_1 \cdot (-i \cdot x_2) + i \cdot x_2 x_1 - i^2 \cdot x_2^2$$

$$= x_1^2 + x_2^2 = |z|^2$$

Polar coordinates:



length: $|z|$

angle: $\varphi \in [0, 2\pi)$

argument of z:
$$\varphi = \begin{cases} \arctan\left(\frac{x_2}{x_1}\right) & , \; x_1 > 0 \; , \; x_2 \geq 0 \\ \frac{\pi}{2} & , \; x_1 = 0 \; , \; x_2 > 0 \\ \arctan\left(\frac{x_2}{x_1}\right) + \pi & , \; x_1 < 0 \\ \frac{3\pi}{2} & , \; x_1 = 0 \; , \; x_2 < 0 \\ \arctan\left(\frac{x_2}{x_1}\right) + 2\pi & , \; x_1 > 0 \; , \; x_2 < 0 \end{cases}$$

$$z = x_1 + i \cdot x_2 = |z| \cdot \left(\cos(\varphi) + i \cdot \sin(\varphi) \right)$$

Example: $z = 3 + i\cdot 3$, $\overline{z} = 3 - i\cdot 3$, $z\cdot\overline{z} = 9 + 9 = 18$

$$\Rightarrow |z| = \sqrt{18} = 3\cdot\sqrt{2} \quad , \quad \varphi = \arctan\left(\frac{3}{3}\right) = \frac{\pi}{4}$$

$$\Rightarrow z = 3\cdot\sqrt{2}\cdot\left(\cos\left(\frac{\pi}{4}\right) + i\cdot\sin\left(\frac{\pi}{4}\right)\right) \overset{\text{later}}{=} 3\cdot\sqrt{2}\cdot e^{i\frac{\pi}{4}}$$