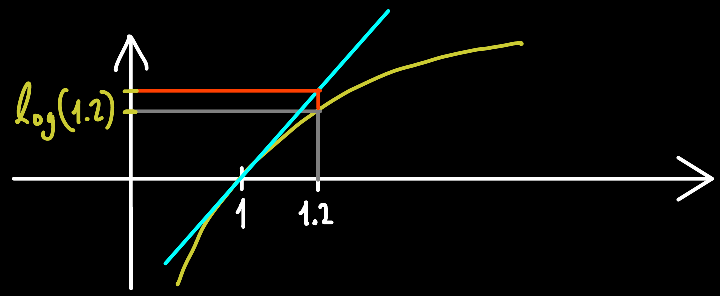
Real Analysis - Part 47

Taylor: $f(x_0+h) = T_n(h) + R_n(h)$

$$T_n(h) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} \cdot h^k}_{n\text{-th order Taylor polynomial}} \qquad R_n(h) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \cdot h^{n+1}$$

ξ between x_0 and x_0+h

Example: $\log(1.2) = ?$



expansion point $x_0 = 1$

$h = 0.2$

$$\log(x) \qquad \log'(x) = \frac{1}{x} \qquad \log''(x) = -\frac{1}{x^2} \qquad \log'''(x) = \frac{2}{x^3} \qquad \log^{(4)}(x) = -\frac{3!}{x^4}$$

$$\log(x_0) = 0 \qquad \log'(x_0) = 1 \qquad \log''(x_0) = -1 \qquad \log'''(x_0) = 2$$

Third order Taylor polynomial: $T_3(h) = 0 \cdot h^0 + \frac{1}{1!} h^1 + \frac{-1}{2!} h^2 + \frac{2}{3!} h^3$

$$= h - \frac{1}{2} h^2 + \frac{1}{3} h^3$$

$$T_3(0.2) = \frac{1}{5} - \frac{1}{2}\left(\frac{1}{5}\right)^2 + \frac{1}{3}\left(\frac{1}{5}\right)^3 = \frac{137}{750} = 0.182\overline{6}$$

first digits of $\log(1.2)$?

$$\left|\log(1.2) - T_3(0.2)\right| = \left|R_3(0.2)\right| = \left|\frac{f^{(3+1)}(\xi)}{(3+1)!} \cdot 0.2^{3+1}\right|$$

$\xi \in (1, 1.2)$

$$= \left|-\frac{3!}{\xi^4} \cdot \frac{1}{4!} \cdot 0.2^4\right| = 4 \cdot 10^{-4} \cdot \frac{1}{\xi^4} \leq 0.0004$$

$$0.182\overline{6} - 0.0004 \leq \log(1.2) \leq 0.182\overline{6} + 0.0004$$

$$0.1822 \leq \log(1.2) \leq 0.1831 \qquad \Rightarrow \quad \log(1.2) = 0.18\ldots$$