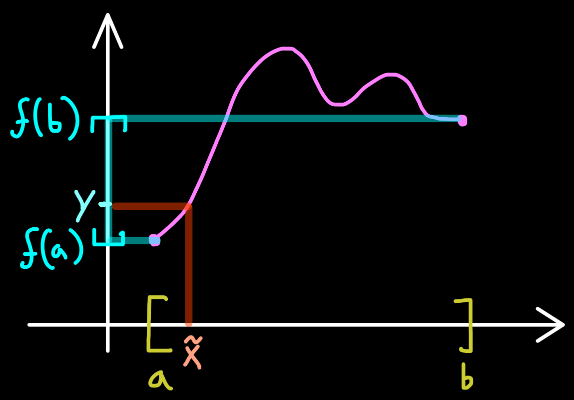
Real Analysis — Part 32

$f: I \longrightarrow \mathbb{R}$ continuous, $I = [a,b]$



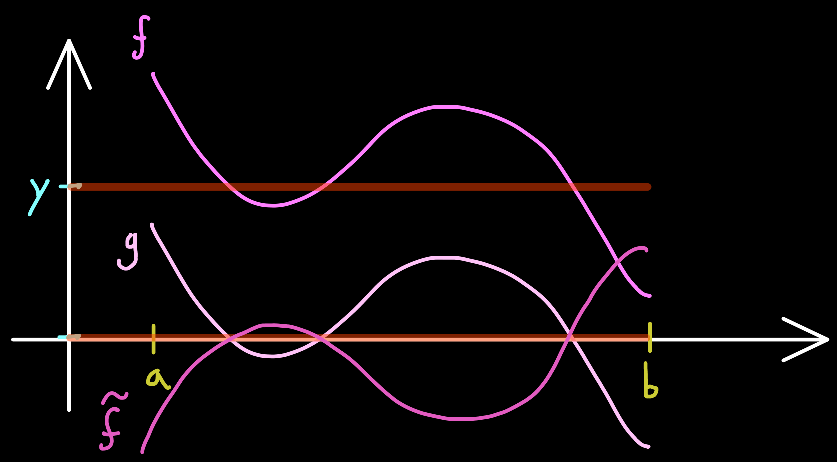
Intermediate value theorem: Let $f: [a,b] \longrightarrow \mathbb{R}$ be continuous and

$$y \in [f(a), f(b)] \quad \text{or} \quad y \in [f(b), f(a)] .$$

Then there is $\tilde{x} \in [a,b]$ with $f(\tilde{x}) = y$.

Corollary: $f\big[[a,b]\big]$ is also an interval.

Proof of the intermediate value theorem:

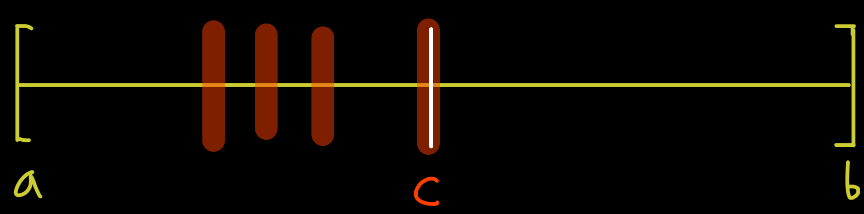


Define new function:

$$g := f - y$$

$$\tilde{f} := \begin{cases} -g & \text{if } g(a) > 0 \\ g & \text{if } g(a) \leq 0 \end{cases}$$

Then $\tilde{f}$ is continuous, $\tilde{y} := 0$, and $\tilde{f}(a) \leq 0$, $\tilde{f}(b) \geq 0$.



Check $\tilde{f}(c)$:

Repeat

If $\tilde{f}(c) > 0$: $b_1 := c$, $a_1 := a$

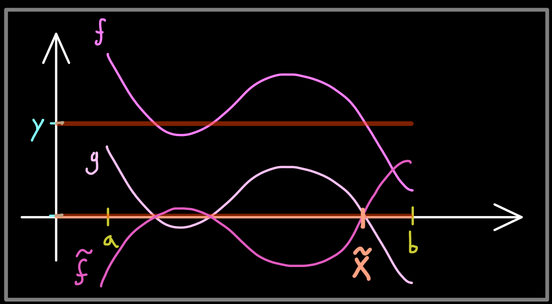
If $\tilde{f}(c) \leq 0$: $a_1 := c$, $b_1 := b$

Wet get two Cauchy sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ and $b_n - a_n \xrightarrow{n\to\infty} 0$

$$\Longrightarrow \quad \tilde{x} := \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n \in [a,b]$$

We know:

$$\begin{matrix} \lim\limits_{n\to\infty} \tilde{f}(a_n) \leq 0 \\ \lim\limits_{n\to\infty} \tilde{f}(b_n) \geq 0 \end{matrix} \quad \Longrightarrow \quad \begin{matrix} \tilde{f}\left(\lim\limits_{n\to\infty} a_n\right) \leq 0 \\ \tilde{f}\left(\lim\limits_{n\to\infty} b_n\right) \geq 0 \end{matrix} \quad \Longrightarrow \quad \begin{matrix} \tilde{f}(\tilde{x}) \leq 0 \\ \tilde{f}(\tilde{x}) \geq 0 \end{matrix}$$



$$\Longrightarrow \quad \tilde{f}(\tilde{x}) = 0 \quad \Longrightarrow \quad \underbrace{g(\tilde{x})}_{= f(\tilde{x}) - y} = 0 \quad \Longrightarrow \quad f(\tilde{x}) = y \qquad \square$$