Exercise 1. Vector space of step functions

Show that the set of step functions $S([a,b])$ *is a real vector space, which means that for any* $\varphi, \psi \in S([a,b])$ *and* λ *we have* $\lambda \cdot \varphi \in S([a,b])$ *and* $\varphi + \psi \in S([a,b])$.

Exercise 2. Properties of step functions

Let $\varphi \in S([a,b])$ *be a step function on the interval* $[a,b]$*. Show the following:*

(a) $\left| \int_a^b \varphi(x)\, dx \right| = \int_a^b |\varphi(x)|\, dx$

(b) $\int_a^b 1\, dx = b - a$

(c) $(b-a) \min_{x \in [a,b]} \varphi(x) \leq \int_a^b \varphi(x)\, dx \leq (b-a) \max_{x \in [a,b]} \varphi(x)$

(d) *For any* c *with* $a < c < b$*, we get:*

$$\int_a^b \varphi(x)\, dx = \int_a^c \varphi(x)\, dx + \int_c^b \varphi(x)\, dx$$

Exercise 3. Pointwise limit of step functions

Give an explicit example of a sequence of step functions (φ_n) *in* $S([a,b])$ *that converge pointwisely but where the pointwise limit is not a step function anymore.*