Open, Closed and Compact Sets

Next, we define some particular sets and special properties of sets. The following definitions hold for $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Hence, we formulate them for real numbers and complex numbers in the same way.

Definition 1. ε-neighbourhood

Let $x \in \mathbb{F}$. Then for $\varepsilon > 0$, the ε-neighbourhood of x is defined by the set

$$B_\varepsilon(x) = \{y \in \mathbb{F} : |x - y| < \varepsilon\}.$$

A set $M \subset \mathbb{F}$ is called neighbourhood of x, if there exists some $\varepsilon > 0$ such that

$$B_\varepsilon(x) \subset M.$$

Example 2. (a) If $\mathbb{F} = \mathbb{R}$, then the ε-neighbourhood of $x \in \mathbb{R}$ is given by the interval

$$B_\varepsilon(x) = (x - \varepsilon, x + \varepsilon).$$

(b) If $\mathbb{F} = \mathbb{C}$, $\varepsilon > 0$, then the ε-neighbourhood of $x \in \mathbb{C}$ consists of all complex numbers being in the interior of a circle in the complex plane with midpoint x and radius ε.

(c) $[0, 1]$ is a neighbourhood of $\frac{1}{2}$ (also of $\frac{3}{4}$, $\frac{1}{\sqrt{2}}$ etc.), but it is not a neighbourhood of 0 or 1.

Definition 3. Open, closed, compact sets

Let $M \subset \mathbb{F}$. Then M is called

(i) open if for all $x \in M$ holds: M is a neighbourhood of x.

(ii) closed if for all convergent sequences $(a_n)_{n\in\mathbb{N}}$ with $a_n \in M$ for all $n \in \mathbb{N}$ holds: $\lim_{n\to\infty} a_n = a \in M$.

(iii) compact if for all sequences $(a_n)_{n\in\mathbb{N}}$ with $a_n \in M$ for all $n \in \mathbb{N}$ holds: There exists some convergent subsequence $(a_{n_k})_{k\in\mathbb{N}}$ with $\lim_{k\to\infty} a_{n_k} = a \in M$.

Example 4. (a) The interval $(0, 1)$ is open.
Proof: Consider $x \in (0, 1)$. Then for $\varepsilon = \min\{x, 1 - x\}$ holds $\varepsilon > 0$ and $B_\varepsilon(x) = (x - \varepsilon, x + \varepsilon) \subset (0, 1)$. □

(b) The interval $(0, 1)$ is not closed.
Proof: Consider the sequence $(a_n)_{n\in\mathbb{N}} = (\frac{1}{n+1})_{n\in\mathbb{N}}$. Clearly, for all $n \in \mathbb{N}$ holds $a_n = \frac{1}{n+1} \in (0, 1)$, but $(a_n)_{n\in\mathbb{N}}$ converges to $0 \notin (0, 1)$. □

(c) The interval $(0, 1)$ is not compact.
Proof: Again consider the sequence $(a_n)_{n\in\mathbb{N}} = (\frac{1}{n+1})_{n\in\mathbb{N}}$ in $(0, 1)$. The convergence of $(a_n)_{n\in\mathbb{N}}$ to $0 \notin (0, 1)$ also implies that this holds true for any subsequence $(a_{n_k})_{k\in\mathbb{N}}$. Hence, any subsequence of the above constructed one is not convergent to some value in $(0, 1)$. □

(d) The interval $(0,1]$ is neither open nor closed.
Proof: The closedness can be disproved by considering again the sequence $(a_n)_{n\in\mathbb{N}} = (\frac{1}{n+1})_{n\in\mathbb{N}}$, whereas the non-openness follows from the fact that $(0,1]$ is not a neighbourhood of 1. □

(e) The set $\mathbb{R}$ is open and closed but not compact.
Proof: Openness and closedness are easy to verify. To see that this set is not compact, consider the sequence $(a_n)_{n\in\mathbb{N}} = (n)_{n\in\mathbb{N}}$ (which is of course in $\mathbb{R}$). It can be readily verified that any subsequence $(a_{n_k})_{k\in\mathbb{N}} = (n_k)_{k\in\mathbb{N}}$ is unbounded, too. Therefore, arbitrary subsequences $(a_{n_k})_{k\in\mathbb{N}} = (n_k)_{k\in\mathbb{N}}$ cannot converge. □

(f) The empty set $\varnothing$ is open, closed and compact.
Proof: $\varnothing$ is a neighbourhood of all $x \in \varnothing$ (there is none, but the statement "for all $x \in \varnothing$" holds then true more than ever). By the same kind of argumentation, we can show that this set is compact and closed. The non-existence of a sequence in $\varnothing$ implies that every statement holds true for them. In particular, all sequences $(a_n)_{n\in\mathbb{N}}$ in $\varnothing$ converge to some $x \in \varnothing$ and have a convergent subsequence with limit in $\varnothing$. □

Next we relate these three concepts to each other.

Theorem 5.

For a set $C \subset \mathbb{F}$, the following statements are equivalent:

(i) C is open;

(ii) $\mathbb{F}\backslash C$ is closed.

Proof:
"(i)⇒(ii)": Let C be open. Consider a convergent sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n \in \mathbb{F}\backslash C$. We have to show that for $a = \lim_{n\to\infty} a_n$ holds $a \in \mathbb{F}\backslash C$. Assume the converse, i.e., $a \in C$. Since C is open, we have that $B_\varepsilon(a) \subset C$ for some $\varepsilon > 0$. By the definition of convergence, there exists some N such that for all $n \geq N$ holds $|a - a_n| < \varepsilon$, i.e.,

$$a_n \in B_\varepsilon(a) \subset C.$$

However, this is a contradiction to $a_n \in \mathbb{F}\backslash C$.

"(ii)⇒(i)": Let $\mathbb{F}\backslash C$ be closed. We have to show that C is open. Assume the converse, i.e., C is not open. In particular, this means that there exists some $a \in C$ such that for all $n \in \mathbb{N}$ holds $B_{\frac{1}{n}}(a) \not\subset C$. This means that for all $n \in \mathbb{N}$, we can find some $a_n \in \mathbb{F}\backslash C$ with $a_n \in B_{\frac{1}{n}}(a)$, i.e., $|a - a_n| < \frac{1}{n}$. As a consequence, for the sequence $(a_n)_{n\in\mathbb{N}}$ holds that

$$\lim_{n\to\infty} a_n = a \in C,$$

but $a_n \in \mathbb{F}\backslash C$ for all $n \in \mathbb{N}$. This is a contradiction to the closedness of $\mathbb{F}\backslash C$. □

Now we present the connection between compactness, closedness and boundedness of subsets of $\mathbb{F}$. Note that these results hold as well in the Euclidean spaces $\mathbb{R}^n$ and $\mathbb{C}^n$.

Exercise 6. Properties of sets

Categorise the following sets in terms of open, closed and compact:

(a) $[a, \infty) \subset \mathbb{R}$ *with* $a \in \mathbb{R}$ *as subset of* $\mathbb{R}$,

(b) $[a, \infty) \subset \mathbb{R}$ *with* $a \in \mathbb{R}$ *as subset of* $\mathbb{C}$,

(c) $\{x + iy \mid y \leq x^2,\ x \in [-1, 1]\}$ *as subset of* $\mathbb{C}$,

(d) $C := \bigcap_{n=1}^{\infty} C_n$ *as subset of* $\mathbb{R}$ *with* $C_n := \frac{1}{3}C_{n-1} \cup \left(\frac{2}{3} + \frac{1}{3}C_{n-1}\right)$, $C_1 := [0, 1]$

Exercise 7. Open and closed sets

Let $(O_n)_{n\in\mathbb{N}}$ *be a family of open sets and* $(A_n)_{n\in\mathbb{N}}$ *a family of closed sets in* $\mathbb{R}$.

(a) *Show that* $\bigcup_{n=1}^{\infty} O_n$ *is open and* $\bigcap_{n=1}^{\infty} A_n$ *is closed.*

(b) *Provide counterexamples to show that* $\bigcap_{n=1}^{\infty} O_n$ *is in general not open while* $\bigcup_{n=1}^{\infty} A_n$ *is in general not closed.*