BECOME A MEMBER

ON STEADY

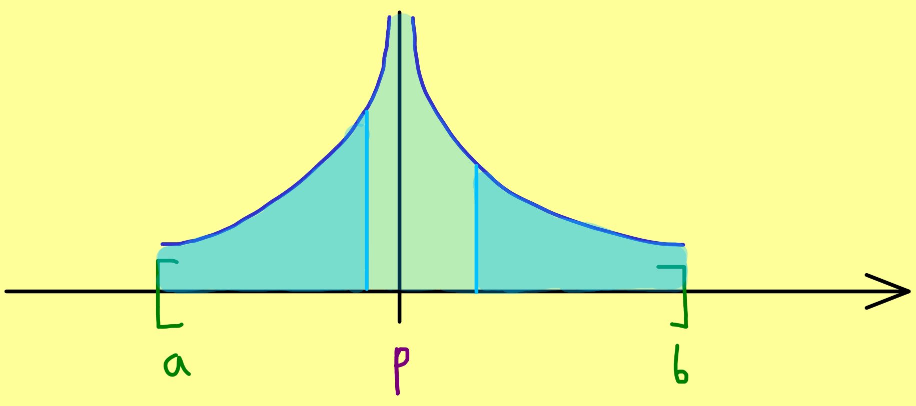
The Bright Side of Mathematics

Real Analysis – Part 64

For $f : [a,b] \setminus \{p\} \longrightarrow \mathbb{R}$

one defines the following

improper Riemann integral:

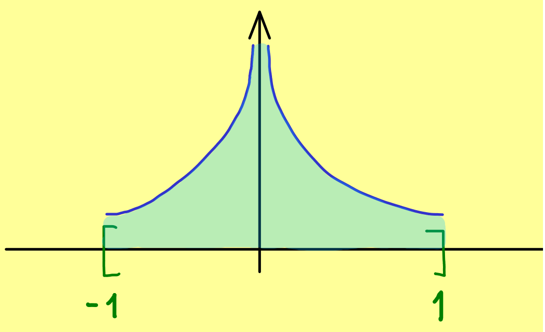


$$\int_a^b f(x)\,dx := \lim_{\varepsilon_1 \searrow 0} \int_a^{p-\varepsilon_1} f(x)\,dx + \lim_{\varepsilon_2 \searrow 0} \int_{p+\varepsilon_2}^{b} f(x)\,dx$$

Example:

$$\int_{-1}^{1} \frac{1}{2\sqrt{|x|}}\,dx = \lim_{\varepsilon_1 \searrow 0} \int_{-1}^{-\varepsilon_1} \frac{1}{2\sqrt{|x|}}\,dx + \lim_{\varepsilon_2 \searrow 0} \int_{\varepsilon_2}^{1} \frac{1}{2\sqrt{|x|}}\,dx$$

$$= \lim_{\varepsilon_1 \searrow 0} \int_{-1}^{-\varepsilon_1} \frac{1}{2\sqrt{-x}}\,dx + \lim_{\varepsilon_2 \searrow 0} \int_{\varepsilon_2}^{1} \frac{1}{2\sqrt{x}}\,dx$$

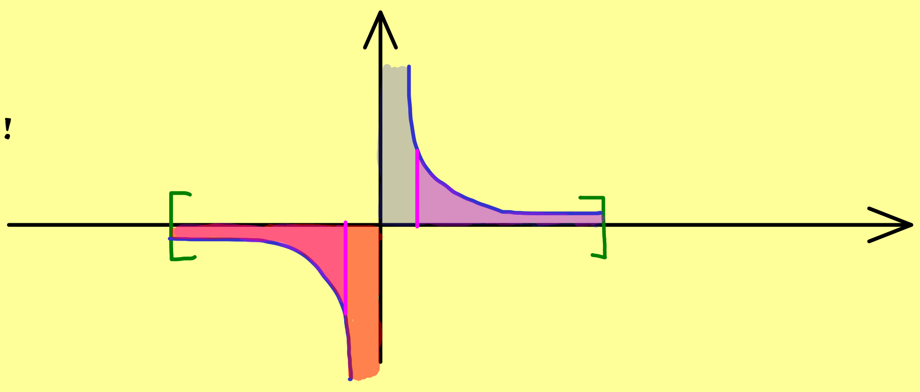


$$= \lim_{\varepsilon_1 \searrow 0} \left(-\sqrt{-x} \,\Big|_{-1}^{-\varepsilon_1} \right) + \lim_{\varepsilon_2 \searrow 0} \left(\sqrt{x} \,\Big|_{\varepsilon_2}^{1} \right)$$

$$= \lim_{\varepsilon_1 \searrow 0} \left(-\sqrt{\varepsilon_1} - \left(-\sqrt{1} \right) \right) + \lim_{\varepsilon_2 \searrow 0} \left(\sqrt{1} - \sqrt{\varepsilon_2} \right) = 2$$

Counterexample:

$$\int_{-1}^{1} \frac{1}{x}\,dx \quad \text{does not exist!}$$



Cauchy principal value:

$$\text{p.v.}\int_{-1}^{1} \frac{1}{x}\,dx := \lim_{\varepsilon \searrow 0} \left(\int_{-1}^{-\varepsilon} \frac{1}{x}\,dx + \int_{\varepsilon}^{1} \frac{1}{x}\,dx \right)$$

$$= \lim_{\varepsilon \searrow 0} \left(\log(|x|) \Big|_{-1}^{-\varepsilon} + \log(|x|) \Big|_{\varepsilon}^{1} \right) = 0$$

$$\text{p.v.}\int_{-\infty}^{\infty} x\,dx = \lim_{a \to \infty} \int_{-a}^{a} x\,dx = 0$$