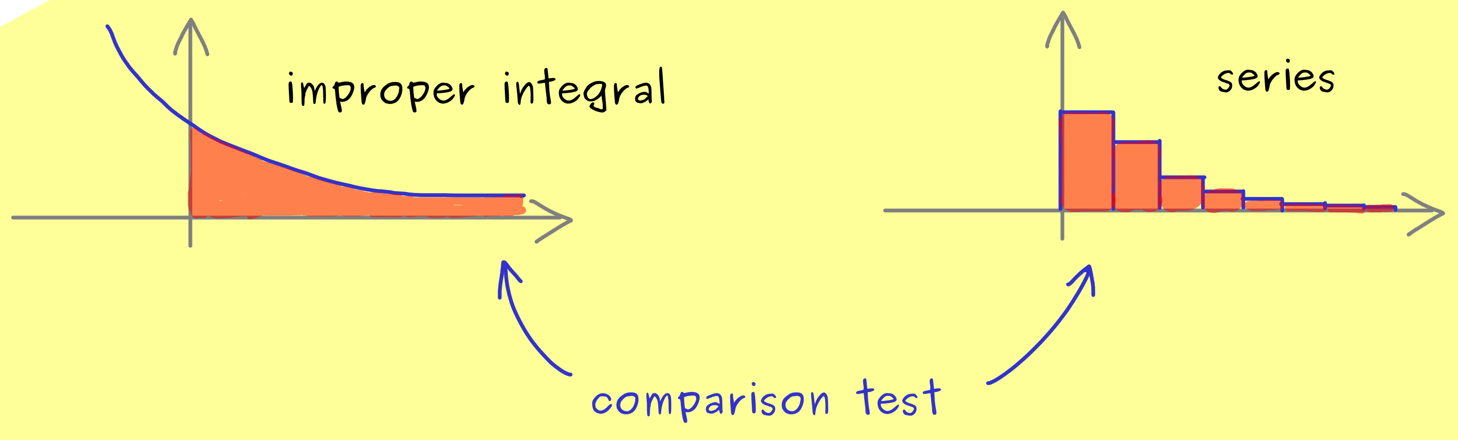
The Bright Side of Mathematics

Real Analysis – Part 61



Theorem: $f, g : [a,\infty) \longrightarrow \mathbb{R}$ with $g(x) \geq 0$ for all $x \in [a,\infty)$ and:

$g|_{[a,b]}, f|_{[a,b]} \in \mathcal{R}([a,b])$ for all $b \geq a$.

(a) If $|f(x)| \leq g(x)$ for all $x \in [a,\infty)$, then:

$$\int_a^\infty g(x)\,dx \text{ converges} \implies \int_a^\infty f(x)\,dx \text{ converges}$$

(b) If $g(x) \leq f(x)$ for all $x \in [a,\infty)$, then:

$$\int_a^\infty g(x)\,dx \text{ diverges} \implies \int_a^\infty f(x)\,dx \text{ diverges}$$

Example: Recall: $\int_1^\infty \frac{1}{x}\,dx$ diverges since $\int_1^b \frac{1}{x}\,dx = \log(b) \xrightarrow{b\to\infty} \infty$

Is $\int_1^\infty \frac{x}{x^2+1}\,dx$ convergent?

$$x \cdot \left(\frac{x}{x^2+1}\right) = \frac{x^2}{x^2+1} = \frac{1}{1+\frac{1}{x^2}} \xrightarrow{x\to\infty} 1$$

so eventually: $x \cdot \left(\frac{x}{x^2+1}\right) \geq \frac{1}{2}$

there is $R \geq 1$ such that for all $x \geq R$: $\frac{x}{x^2+1} \geq \frac{1}{2} \cdot \frac{1}{x}$

$\int_R^\infty \frac{x}{x^2+1}\,dx$ is divergent because $\int_R^\infty \frac{1}{2} \cdot \frac{1}{x}\,dx$ is divergent