The Bright Side of Mathematics

Real Analysis – Part 38

Examples: (a) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x$ $\Rightarrow$ $f': \mathbb{R} \to \mathbb{R}$, $f'(x) = 1$

(b) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 = x \cdot x$

product rule: $f'(x) = x \cdot 1 + 1 \cdot x = 2 \cdot x$

(c) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 = x^2 \cdot x$

product rule: $f'(x) = x^2 \cdot 1 + 2 \cdot x \cdot x = 3 \cdot x^2$

(d) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^n$, $n \in \mathbb{N}$

$f'(x) = n \cdot x^{n-1}$ (proof by induction + product rule)

(e) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = a_m \cdot x^m + a_{m-1} x^{m-1} + \dots + a_1 \cdot x^1 + a_0$

$f'(x) = a_m \cdot m \cdot x^{m-1} + a_{m-1} (m-1) \cdot x^{m-2} + \dots + a_1$

(f) power series: $f(x) = \sum_{k=0}^{\infty} a_k \cdot x^k$, $f'(x) = \sum_{k=1}^{\infty} a_k \cdot k \cdot x^{k-1}$?

General result for power series: Let $f: (-r, r) \to \mathbb{R}$, $f(x) = \sum_{k=0}^{\infty} a_k \cdot x^k$, be a power series with radius of convergence $r > 0$.

(1) $\sum_{k=0}^{\infty} a_k \cdot x^k$ is uniformly convergent on each interval $[-c, c] \subseteq (-r, r)$

(sequence of functions $g_n: [-c, c] \to \mathbb{R}$, $g_n(x) = \sum_{k=0}^{n} a_k \cdot x^k$ is uniformly convergent)

(2) $\sum_{k=1}^{\infty} a_k \cdot k \cdot x^{k-1}$ is uniformly convergent on each interval $[-c, c] \subseteq (-r, r)$

(sequence of functions $g_n': [-c, c] \to \mathbb{R}$, $g_n'(x) = \sum_{k=1}^{n} a_k \cdot k \cdot x^{k-1}$ is uniformly convergent)

(3) $f'(x) = \sum_{k=1}^{\infty} a_k \cdot k \cdot x^{k-1}$

Proof: (1) $\underbrace{\| f - g_n \|_\infty}_{\text{supremum norm on } [-c,c]} = \left\| \sum_{k=n+1}^{\infty} a_k \cdot x^k \right\|_\infty = \sup_{x \in [-c,c]} \lim_{N \to \infty} \left| \sum_{k=n+1}^{N} a_k \cdot x^k \right|$

$\overset{\Delta\text{-inequality}}{\leq} \sup_{x \in [-c,c]} \lim_{N \to \infty} \sum_{k=n+1}^{N} |a_k| \cdot \underbrace{|x|^k}_{\leq c^k} \leq \sum_{k=n+1}^{\infty} |a_k| \cdot c^k \overset{(*)}{\leq} \underbrace{\mathcal{B}}_{\text{constant}} \cdot \sum_{k=n+1}^{\infty} q^k \xrightarrow{n \to \infty} 0$, $|q| < 1$

(*) By assumption $\sum_{k=0}^{\infty} a_k \cdot \tilde{r}^k$ is convergent for $c < \tilde{r} < r$.

Hence there is $\mathcal{B}$ with $\mathcal{B} \geq |a_k \cdot \tilde{r}^k| = |a_k| \cdot \tilde{r}^k = |a_k| \cdot c^k \cdot \Big(\underbrace{\frac{\tilde{r}}{c}}_{q^{-1}} \Big)^k$

$\Rightarrow \mathcal{B} \cdot q^k \geq |a_k| \cdot c^k$

(2) Same proof as in (1) because the radius of convergence is the same.

(3) Pointwise convergence of functions + uniform convergence of derivatives:

$\overset{\text{part 37}}{\Longrightarrow}$ f differentiable and $f'(x) = \sum_{k=1}^{\infty} a_k \cdot k \cdot x^{k-1}$

Examples: (a) $\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \Rightarrow \exp'(x) = \sum_{k=1}^{\infty} \frac{1}{k!} \cdot k \cdot x^{k-1} = \sum_{k=1}^{\infty} \frac{1}{(k-1)!} \cdot x^{\overset{\text{new } k}{k-1}}$

$= \sum_{k=0}^{\infty} \frac{x^k}{k!} = \exp(x)$

(b) $\sin(x) = \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+1}}{(2m+1)!} \Rightarrow \sin'(x) = \sum_{m=0}^{\infty} (-1)^m \frac{1}{(2m+1)!} (2m+1) \cdot x^{2m}$

$= \sum_{m=0}^{\infty} (-1)^m \cdot \frac{1}{(2m)!} x^{2m} = \cos(x)$