The Bright Side of Mathematics

# Real Analysis – Part 37

sequence of functions:

$(f_1, f_2, f_3, f_4, f_5, \dots)$ $\quad f_n : I \to \mathbb{R}$ , $f : I \to \mathbb{R}$

$\boxed{\text{Uniform convergence means: } \|f_n - f\|_\infty \xrightarrow{n\to\infty} 0}$

**Fact:** $f_n$ continuous and $\|f_n - f\|_\infty \xrightarrow{n\to\infty} 0 \implies f$ continuous

**Definition:** $f : I \to \mathbb{R}$ is called <u>differentiable</u> if $f$ is differentiable at all $x_0 \in I$.

In this case, $f' : I \to \mathbb{R}$ defined by $x \mapsto f'(x)$ is called <u>derivative</u> of $f$

**Example:** $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 4 \cdot x + 5 \implies f' : \mathbb{R} \to \mathbb{R}$ , $f'(x) = 4$

**Theorem:** Let $(f_1, f_2, f_3, f_4, f_5, \dots)$ be a sequence of functions $f_n : I \to \mathbb{R}$.

Assume:
- $(f_n)_{n\in\mathbb{N}}$ is pointwisely convergent to a function $f : I \to \mathbb{R}$
- $f_n : I \to \mathbb{R}$ differentiable for all $n \in \mathbb{N}$
- There is $g : I \to \mathbb{R}$ with $\|f_n' - g\|_\infty \xrightarrow{n\to\infty} 0$

Then: $\|f_n - f\|_\infty \xrightarrow{n\to\infty} 0$ and $f$ <u>differentiable</u> with $f' = g$ .

**Proof:** Let $x_0 \in I$.

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - g(x_0) \right| \le \underbrace{\left| \frac{f(x) - f(x_0)}{x - x_0} - \frac{f_n(x) - f_n(x_0)}{x - x_0} \right|} + \underbrace{\left| \frac{f_n(x) - f_n(x_0)}{x - x_0} - f_n'(x_0) \right|}_{\xrightarrow{x \to x_0} 0} + \underbrace{\left| f_n'(x_0) - g(x_0) \right|}_{\xrightarrow{n\to\infty} 0}$$

needs

$\|f_n' - g\|_\infty \xrightarrow{n\to\infty} 0$

mean value theorem is helpful
(see later video!)

For any $\varepsilon > 0$:

$$\left| \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} - g(x_0) \right| \le \varepsilon$$