BECOME A MEMBER
ON STEADY

The Bright Side of Mathematics

Real Analysis – Part 19

$\sum_{k=1}^{\infty} a_k$ is called absolutely convergent if $\sum_{k=1}^{\infty} |a_k|$ is convergent.

abs. convergent $\Rightarrow$ convergent: $\sum_{k=1}^{\infty} |a_k|$ is convergent $\Rightarrow$

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n \geq m \geq N: \sum_{k=m}^{n} |a_k| < \varepsilon \quad \text{(Cauchy criterion)}$$

$$\Rightarrow \forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n \geq m \geq N: \left| \sum_{k=m}^{n} a_k \right| \leq \sum_{k=m}^{n} |a_k| < \varepsilon$$

$$\overset{\text{(Cauchy criterion)}}{\Rightarrow} \sum_{k=1}^{\infty} a_k \text{ is convergent}$$

Counterexample: $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$ is convergent but not absolutely convergent
(Leibniz criterion) (harmonic series)

Majorant criterion: Let $\sum_{k=1}^{\infty} a_k$ be a series.

If there is $n_0 \in \mathbb{N}$ and a convergent series $\sum_{k=1}^{\infty} b_k$ with $b_k \geq 0$ and with $|a_k| \leq b_k$ for all $k \geq n_0$, then $\sum_{k=1}^{\infty} a_k$ is abs. convergent.

Proof: Apply Cauchy criterion to $\sum_{k=1}^{\infty} b_k$:

$$\forall \varepsilon > 0 \ \exists N \geq n_0 \ \forall n \geq m \geq N: \sum_{k=m}^{n} |a_k| \leq \sum_{k=m}^{n} b_k = \left| \sum_{k=m}^{n} b_k \right| < \varepsilon$$

Minorant criterion: Let $\sum_{k=1}^{\infty} a_k$ be a series with $a_k \geq 0$.

If there is $n_0 \in \mathbb{N}$ and a divergent series $\sum_{k=1}^{\infty} b_k$ with $b_k \geq 0$ and with $a_k \geq b_k$ for all $k \geq n_0$, then $\sum_{k=1}^{\infty} a_k$ is divergent.

Example: $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ is divergent because $\sqrt{k} \leq k \Leftrightarrow \frac{1}{\sqrt{k}} \geq \frac{1}{k}$ for all $k \geq 1$

and $\sum_{k=1}^{\infty} \frac{1}{k}$ is divergent