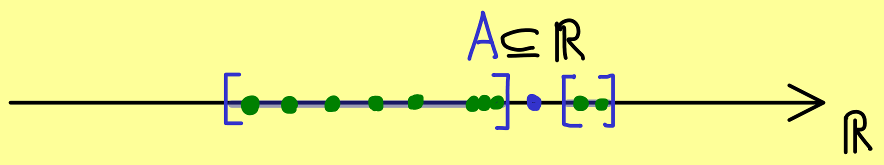
The Bright Side of Mathematics

Real Analysis – Part 14

Compact set (sequentially compact set):



Any sequence $(a_n)_{n\in\mathbb{N}} \subseteq A$ has an accumulation value $a \in A$.

Example: (a) $\emptyset$ is compact.

(b) $\{5\}$ is compact.

(c) $\mathbb{R}$ is not compact. $(a_n)_{n\in\mathbb{N}} = (n)_{n\in\mathbb{N}}$ has no accumulation value $a \in \mathbb{R}$

(d) $[c,d]$, $c \leq d$, compact set.

Let $(a_n)_{n\in\mathbb{N}} \subseteq [c,d] \Longrightarrow (a_n)_{n\in\mathbb{N}}$ is bounded

$\overset{\text{Bolzano–Weierstrass theorem}}{\Longrightarrow}$ $(a_n)_{n\in\mathbb{N}}$ has an accumulation value $a \in \mathbb{R}$

$\overset{[c,d] \text{ closed}}{\Longrightarrow}$ accumulation value actually satisfies $a \in [c,d]$

Heine–Borel theorem For $A \subseteq \mathbb{R}$, we have:

A is compact $\Longleftrightarrow$ A is bounded and closed

Proof: ($\Longleftarrow$) Do the same as before with Bolzano–Weierstrass theorem.

($\Longrightarrow$) Assume A is compact.

Let $(a_n)_{n\in\mathbb{N}} \subseteq A$ be a convergent sequence with limit $\tilde{a} \in \mathbb{R}$.

$\overset{A \text{ is compact}}{\Longrightarrow}$ $(a_n)_{n\in\mathbb{N}}$ has an accumulation value $a \in A$.

$\overset{\text{only one acc. value}}{\Longrightarrow}$ $\tilde{a} = a \in A$ $\Longrightarrow$ A is closed.

Assume A is not bounded.

$\Longrightarrow$ There is a sequence $(a_n)_{n\in\mathbb{N}} \subseteq A$ with $|a_n| > n$ for all $n \in \mathbb{N}$.

$\Longrightarrow$ no accumulation value $\Longrightarrow$ A is not compact.