The Bright Side of Mathematics

Real Analysis – Part 13

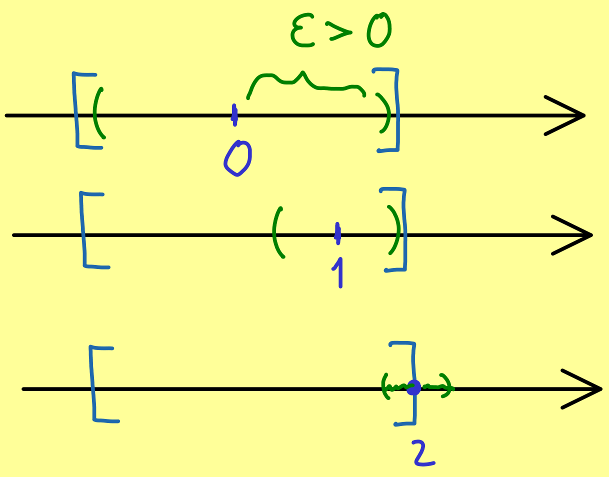
For $\varepsilon > 0$: $(x-\varepsilon, x+\varepsilon) =: B_\varepsilon(x)$ ε-neighbourhood of x

$M \subseteq \mathbb{R}$ is called a neighbourhood of x if there is $\varepsilon > 0$ such that $M \supseteq B_\varepsilon(x)$

Example: $[-2,2]$ is a neighbourhood of 0

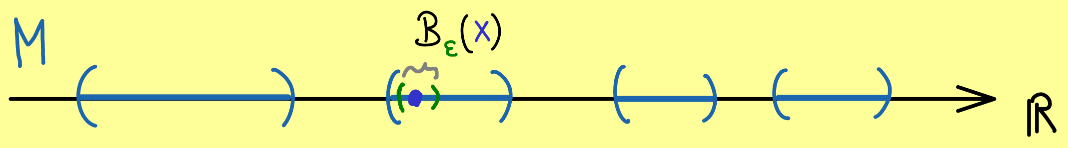
$[-2,2]$ is a neighbourhood of 1

$[-2,2]$ is not a neighbourhood of 2



Definition: $M \subseteq \mathbb{R}$ is called open (in $\mathbb{R}$) if, for all $x \in M$, M is a neighbourhood of x.

$$\forall x \in M \quad \exists \varepsilon > 0 : \quad M \supseteq B_\varepsilon(x)$$

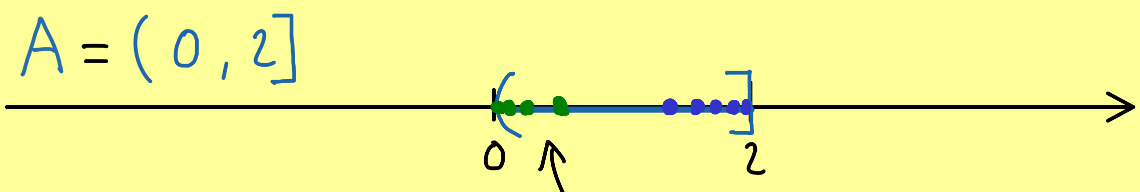


$A \subseteq \mathbb{R}$ is called closed (in $\mathbb{R}$) if $A^c := \mathbb{R} \setminus A$ is open.

Example:
- $\emptyset$, $\mathbb{R}$ are both open and closed.
- $[-2,2]$ is closed but not open.
- $(-2,2)$ is open but not closed.
- $(-2,2]$ is neither open nor closed.

Fact: $A \subseteq \mathbb{R}$ is closed $\Longleftrightarrow$ For all convergent sequences $(a_n)_{n\in\mathbb{N}}$ with $a_n \in A$ for all $n \in \mathbb{N}$, we have: $\lim\limits_{n\to\infty} a_n \in A$

Example: $A = (0,2]$



Take $(a_n)_{n\in\mathbb{N}} = \left(\frac{1}{n}\right)_{n\in\mathbb{N}}$.

Then: $\lim\limits_{n\to\infty} a_n = 0 \notin (0,2]$

Definition: $A \subseteq \mathbb{R}$ is called compact if for all sequences $(a_n)_{n\in\mathbb{N}}$ with $a_n \in A$ for all $n\in\mathbb{N}$, there is a convergent subsequence $(a_{n_k})_{k\in\mathbb{N}}$ with $\lim\limits_{k\to\infty} a_{n_k} \in A$.