The Bright Side of Mathematics

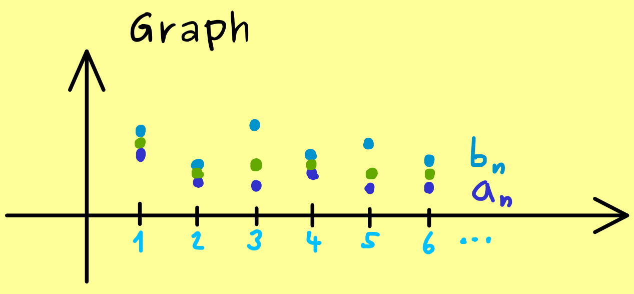
Real Analysis – Part 5

$(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ convergent sequences.

$$\Rightarrow \lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n \,, \quad \lim_{n\to\infty}(a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

In particular: $\lim_{n\to\infty}(a \cdot b_n) = a \cdot \lim_{n\to\infty}(b_n)$

Graph



Properties: $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ convergent sequences.

(a) Monotonicity: $a_n \leq b_n$ for all $n \in \mathbb{N}$

$$\Rightarrow \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$$

(b) Sandwich theorem: $a_n \leq c_n \leq b_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$

$$\Rightarrow (c_n)_{n\in\mathbb{N}} \text{ convergent with } \lim_{n\to\infty} c_n = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

Proof of (b): $(b_n - a_n) \xrightarrow{n\to\infty} \underbrace{\lim_{n\to\infty} a_n}_{a} - \underbrace{\lim_{n\to\infty} b_n}_{b} = 0$ (by the limit theorems)

$d_n := c_n - a_n \quad \Rightarrow \quad 0 \leq d_n \leq b_n - a_n$

Let $\varepsilon > 0$. Then there is $N \in \mathbb{N}$ with: $\forall n \geq N: \ |b_n - a_n| < \varepsilon$

$$|d_n - 0| \leq$$

$\Rightarrow (d_n)_{n\in\mathbb{N}}$ is convergent with limit 0

limit theorems

$\Rightarrow (c_n)_{n\in\mathbb{N}} = (d_n + a_n)_{n\in\mathbb{N}}$ is convergent with limit a $\square$

Example: sequence $(c_n)_{n\in\mathbb{N}}$ given by $c_n = \sqrt{n^2+1} - n$ convergent? limit?

$$= \left(\sqrt{n^2+1} - n\right) \cdot \frac{\left(\sqrt{n^2+1} + n\right)}{\sqrt{n^2+1} + n}$$

$$= \frac{n^2 + 1 - n^2}{\sqrt{n^2+1} + n} = \frac{1}{\underbrace{\sqrt{n^2+1}}_{>0} + n} \leq \frac{1}{n}$$

$$\Rightarrow 0 \leq c_n \leq \frac{1}{n} \text{ for all } n \in \mathbb{N} \underset{\text{Sandwich}}{\Rightarrow} \lim_{n\to\infty} c_n = 0$$