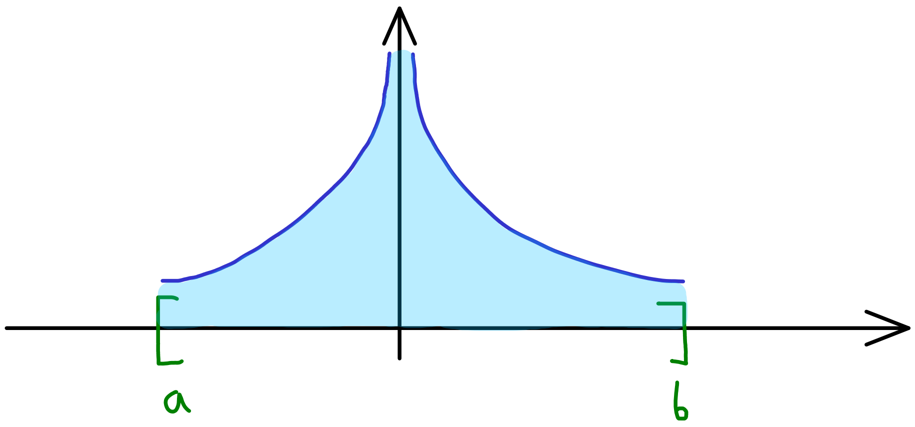
Real Analysis - Part 63

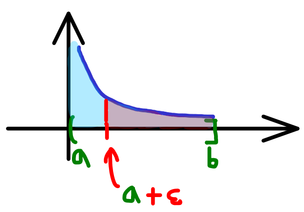
We know: $f: [a,b] \longrightarrow \mathbb{R}$ Riemann-integrable

$\Longrightarrow$ f is bounded

What about this?



Definition: Let $f: (a,b] \longrightarrow \mathbb{R}$ be a function with the property that



$$f\big|_{[a+\varepsilon, b]} \in \mathcal{R}\big([a+\varepsilon, b]\big) \quad \text{for all} \quad \varepsilon > 0 .$$

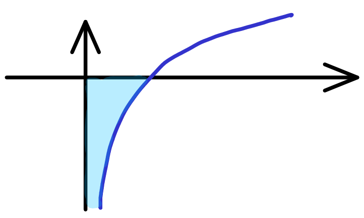
If $\lim\limits_{\varepsilon \searrow 0} \int_{a+\varepsilon}^{b} f(x)\, dx$ exists, we write $\int_a^b f(x)\, dx$ for this limit and we say the integral converges.

Example: $\int_0^1 \log(x)\, dx$?

$$\boxed{\begin{aligned} f'(x) &= 1 \\ g(x) &= \log(x) \\ f(x) &= x \\ g'(x) &= \tfrac{1}{x} \end{aligned}}$$

$$\int_\varepsilon^1 \log(x)\, dx = \int_\varepsilon^1 1 \cdot \log(x)\, dx \overset{\substack{\text{integration} \\ \text{by parts}}}{=} x \cdot \log(x)\Big|_\varepsilon^1 - \int_\varepsilon^1 x \cdot \frac{1}{x}\, dx = x \cdot \big(\log(x) - 1\big)\Big|_\varepsilon^1$$

$$\lim_{\varepsilon \searrow 0} \int_\varepsilon^1 \log(x)\, dx = 1 \cdot \big(\log(1) - 1\big) - \lim_{\varepsilon \searrow 0} \varepsilon \cdot \big(\log(\varepsilon) - 1\big)$$



$$= -1 - \lim_{\varepsilon \searrow 0} \varepsilon \cdot \log(\varepsilon) = -1 - \lim_{\varepsilon \searrow 0} \frac{\log(\varepsilon)}{\frac{1}{\varepsilon}} = -1$$