The Bright Side of Mathematics

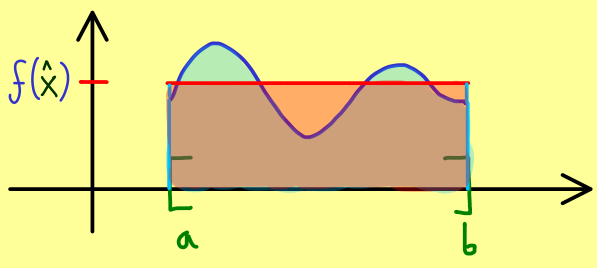
Real Analysis – Part 56

Mean value theorem of integration

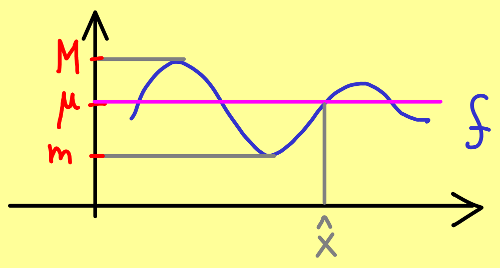
$f, g : [a,b] \to \mathbb{R}$ continuous , $g \geq 0$.

Then there is $\hat{x} \in [a,b]$ with $\int_a^b f(x) g(x)\, dx = f(\hat{x}) \cdot \int_a^b g(x)\, dx$

(often: $g = 1$: $\int_a^b f(x)\, dx = f(\hat{x}) \cdot (b-a)$)



Proof:



$m \leq f(x) \leq M$ for all $x \in [a,b]$

$g \geq 0 \Rightarrow m\, g(x) \leq f(x) g(x) \leq M\, g(x)$

monotonicity: $m \int_a^b g(x)\, dx \leq \int_a^b f(x) g(x)\, dx \leq M \int_a^b g(x)\, dx$

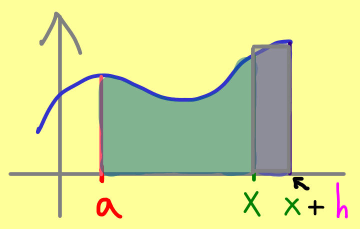
there is $\mu \in [m, M]$: $\mu \int_a^b g(x)\, dx = \int_a^b f(x) g(x)\, dx$

$\xRightarrow{\text{intermediate value theorem}}$ there is $\hat{x} \in [a,b]$ with $f(\hat{x}) = \mu$ $\square$

Proof of the first fundamental theorem of calculus:

$F(x) := \int_a^x f(t)\, dt$

$F(x+h) - F(x) = \int_x^{x+h} f(t)\, dt$



$= f(\hat{x}) \cdot h$ with $\hat{x} \in [x, x+h]$ (or $\hat{x} \in [x+h, x]$)

$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} f(\hat{x}) = f(x) \Rightarrow F' = f$ $\square$

Proof of the second fundamental theorem of calculus:

$\boxed{\int_a^b f(t)\, dt = F(b) - F(a)}$

holds for F_0 ✓

$F_0(x) := \int_a^x f(t)\, dt$ antiderivative of f with $F_0(a) = 0$

arbitrary antiderivative of f: $F = F_0 + c$ for $c \in \mathbb{R}$

$F(b) - F(a) = F_0(b) - F_0(a) = \int_a^b f(t)\, dt$ $\square$