The Bright Side of Mathematics

Real Analysis – Part 33

① Exponential function

$\exp : \mathbb{R} \to \mathbb{R}$ defined by

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$e := \exp(1)$ Euler's number

$$\exp(1) = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = 2.718\ldots$$

We have shown: $\exp(x+y) = \exp(x) \cdot \exp(y)$

For example: $\exp(2) = \exp(1+1) = \exp(1) \cdot \exp(1) = e^2$

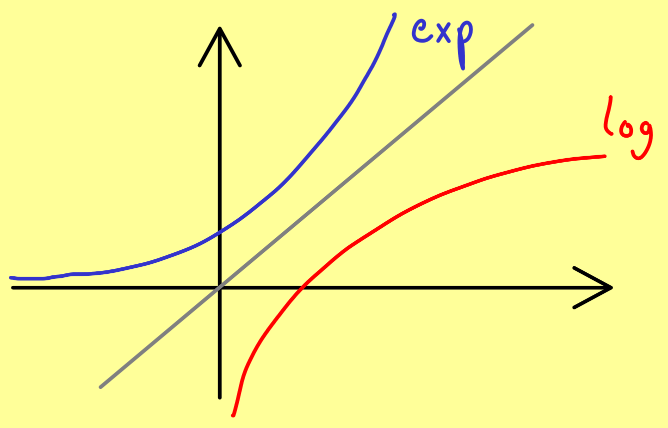
In general: $\exp(x) = e^x$ for $x \in \mathbb{R}$

More properties:

- $\exp$ is a continuous function
- $\exp$ is strictly monotonically increasing $\left(x < y \Rightarrow \exp(x) < \exp(y) \right)$
- $\lim_{x \to \infty} \exp(x) = \infty$, $\lim_{x \to -\infty} \exp(x) = 0$
- $\exp : \mathbb{R} \to (0, \infty)$ is bijective

② Logarithm function

$\log : (0, \infty) \to \mathbb{R}$ defined by the inverse of $\exp : \mathbb{R} \to (0, \infty)$



- $\log$ is a continuous function
- $\log$ is strictly monotonically increasing
- $\log(x \cdot y) = \log(x) + \log(y)$

③ Polynomials

$f : \mathbb{R} \to \mathbb{R}$, $f(x) = \underbrace{a_m}_{\neq 0} \cdot x^m + a_{m-1} x^{m-1} + \cdots + a_1 \cdot x^1 + a_0$ polynomial has degree m

continuous

④ Power series

$f : D \to \mathbb{R}$, $f(x) = \sum_{k=0}^{\infty} a_k \cdot x^k$, $D := \left\{ x \in \mathbb{R} \;\middle|\; \sum_{k=0}^{\infty} a_k \cdot x^k \text{ converges} \right\}$

Example: $(a_k) = \left(0, \frac{1}{1!}, 0, -\frac{1}{3!}, 0, \frac{1}{5!}, 0, -\frac{1}{7!}, 0, \frac{1}{9!}, 0, -\frac{1}{11!}, \ldots\right)$

gives power series $\sin(x) := \sum_{k=0}^{\infty} a_k \cdot x^k$

Theorem: For a power series $\sum_{k=0}^{\infty} a_k \cdot x^k$, there is a maximal $r \in [0, \infty) \cup \{\infty\}$ with $(-r, r) \subseteq D$. It holds: $\limsup_{k \to \infty} \sqrt[k]{|a_k|} = \frac{1}{r}$ $\left(\frac{1}{0} = \infty, \; \frac{1}{\infty} = 0 \right)$

power series is continuous on this interval

(Cauchy–Hadamard)