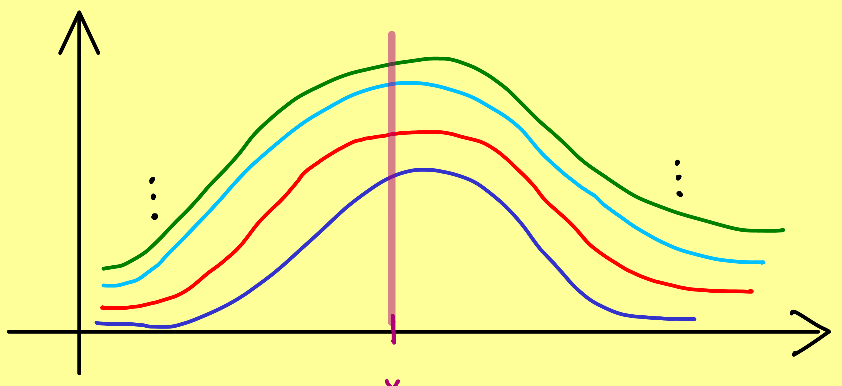
The Bright Side of Mathematics

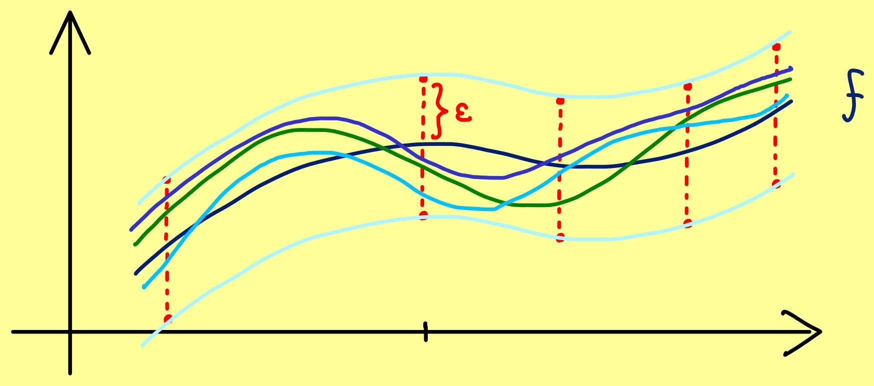
Real Analysis – Part 31

$(f_1, f_2, f_3, f_4, f_5, \dots)$

pointwise convergence:



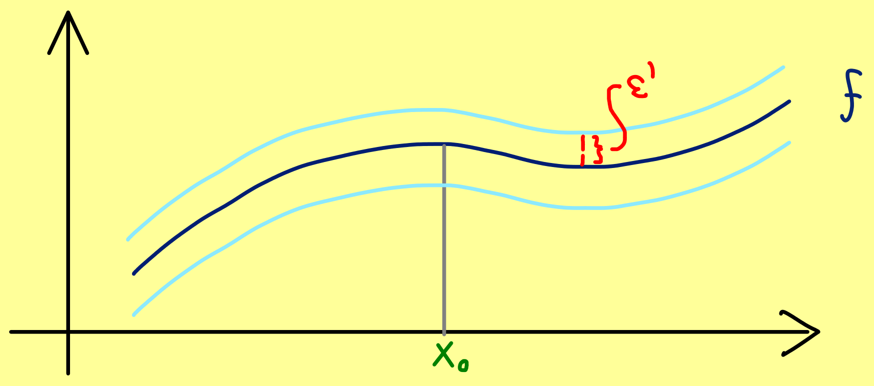
uniform convergence:



Theorem: $I \subseteq \mathbb{R}$, $f_n : I \longrightarrow \mathbb{R}$ continuous (for all $n \in \mathbb{N}$), and $(f_n)_{n \in \mathbb{N}}$ uniformly converges to $f : I \longrightarrow \mathbb{R}$.

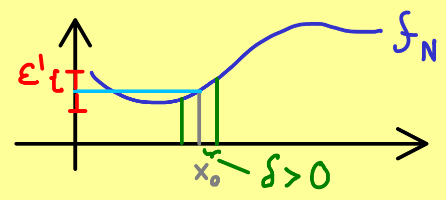
Then: f is also continuous.

Proof: Let $\varepsilon > 0$. Let $x_0 \in I$. Set: $\varepsilon' := \frac{\varepsilon}{3}$ (see end of the proof)



Uniform convergence: $\forall \varepsilon' > 0 \;\; \exists N \in \mathbb{N} \;\; \forall n \geq N \;\; \forall \tilde{x} \in I : \; |f_n(\tilde{x}) - f(\tilde{x})| < \varepsilon'$

Continuity of f_N:



We find $\delta > 0$ with:

$$\forall x \in I : \; |x - x_0| < \delta \implies |f_N(x) - f_N(x_0)| < \varepsilon'$$

Hence:
$$|f(x) - f(x_0)| = |f(x) - f_N(x) + f_N(x) - f_N(x_0) + f_N(x_0) - f(x_0)|$$
$$\leq \underbrace{|f(x) - f_N(x)|}_{< \varepsilon'} + \underbrace{|f_N(x) - f_N(x_0)|}_{< \varepsilon'} + \underbrace{|f_N(x_0) - f(x_0)|}_{< \varepsilon'}$$
$$< 3 \cdot \varepsilon' = \varepsilon$$

Conclusion: We find $\delta > 0$ with: $\forall x \in I : \; |x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$

$\implies f$ is continuous at x_0 $\overset{x_0 \text{ arbitrary}}{\implies}$ f is continuous $\square$