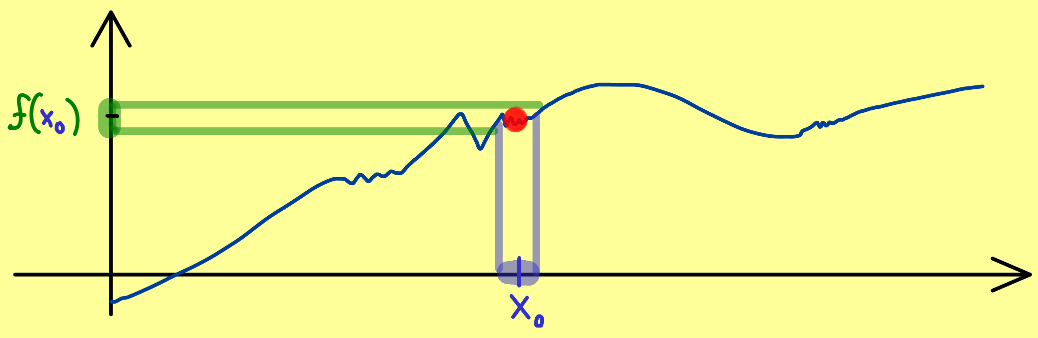
The Bright Side of Mathematics

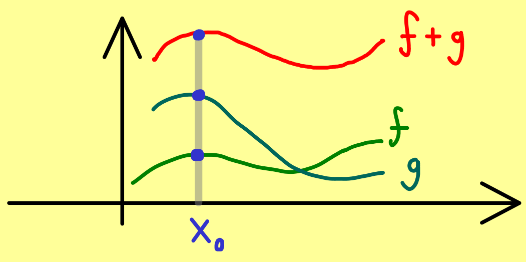
Real Analysis – Part 29



Proposition: $f : I \to \mathbb{R}$, $g : I \to \mathbb{R}$ continuous at $x_0 \in I$,

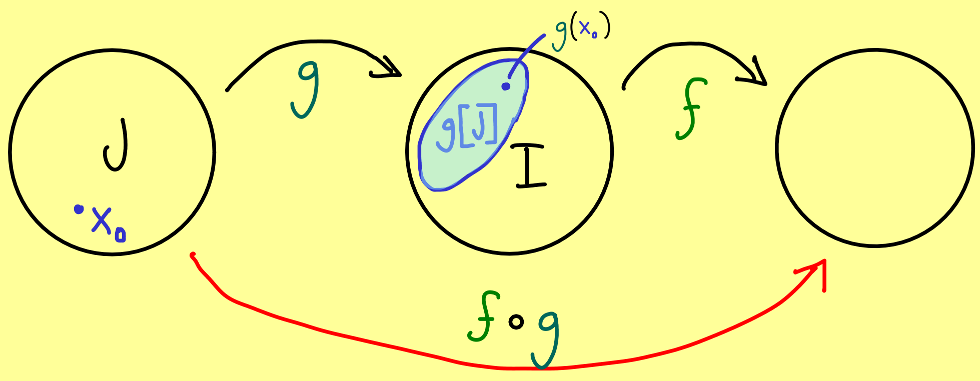
then $f + g : I \to \mathbb{R}$ continuous at $x_0 \in I$,

$f \cdot g : I \to \mathbb{R}$ continuous at $x_0 \in I$.



If in addition $g(x_0) \neq 0$, then $\dfrac{f}{g}$ is continuous at $x_0 \in I$.

Composition of functions:



Proposition: $f : I \to \mathbb{R}$, $g : J \to \mathbb{R}$, $I, J \subseteq \mathbb{R}$, with $g[J] \subseteq I$.

$$\left.\begin{array}{l} g \text{ continuous at } x_0 \in J \\ f \text{ continuous at } g(x_0) \in I \end{array}\right\} \Longrightarrow f \circ g : J \to \mathbb{R} \text{ continuous at } x_0 \in J$$

Proof: Choose sequence $(x_n)_{n \in \mathbb{N}} \subseteq J \setminus \{x_0\}$ with limit x_0.

f is continuos at $g(x_0)$ and $\lim_{n\to\infty} g(x_n) = g(x_0)$

$$\lim_{n\to\infty} (f \circ g)(x_n) = \lim_{n\to\infty} f(g(x_n)) = f\left(\lim_{n\to\infty} g(x_n)\right)$$

g is continuous at x_0

$$= f\left(g\left(\lim_{n\to\infty} x_n\right)\right) = (f \circ g)(x_0)$$