



# The Bright Side of Mathematics

## Real Analysis - Part 21

$$\sum_{k=1}^n a_k = a_1 + a_2 + a_3 + \dots + a_n \quad (n \text{ is even})$$

$$= a_2 + a_1 + a_4 + a_3 + \dots + a_n + a_{n-1}$$

Reordering does not change a finite sum!

Example:  $\sum_{k=0}^{\infty} (-1)^k = 1 + (-1) + 1 + (-1) + 1 + (-1) + 1 + (-1) + \dots$

- is not convergent
- but has two accumulation values 0, 1

a reordering =  $1 + 1 + (-1) + 1 + (-1) + 1 + (-1) + 1 + (-1) + \dots$

- is not convergent
- but has two accumulation values 1, 2

Example:  $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$  convergent by the Leibniz criterion

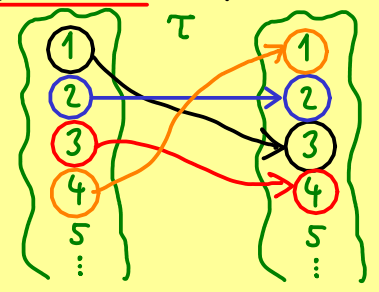
$$= \frac{1}{1} + \left(-\frac{1}{2}\right) + \frac{1}{3} + \left(-\frac{1}{4}\right) + \frac{1}{5} + \left(-\frac{1}{6}\right) + \frac{1}{7} + \dots = c \stackrel{\log(2)}{>} 0$$

a reordering =  $\frac{1}{1} + \frac{1}{3} + \left(-\frac{1}{2}\right) + \frac{1}{5} + \frac{1}{7} + \left(-\frac{1}{4}\right) + \frac{1}{9} + \frac{1}{11} + \left(-\frac{1}{6}\right) + \dots$

$$= \frac{3}{2} \cdot c \quad \text{different limits!}$$

Definition: Let  $\sum_{k=1}^{\infty} a_k$  be a series and  $\tau: \mathbb{N} \rightarrow \mathbb{N}$  be a bijjective map.

Then  $\sum_{k=1}^{\infty} a_{\tau(k)}$  is called a reordering of  $\sum_{k=1}^{\infty} a_k$ .



Theorem: If  $\sum_{k=1}^{\infty} a_k$  is absolutely convergent, then:

(for any  $\tau: \mathbb{N} \rightarrow \mathbb{N}$  bijective)

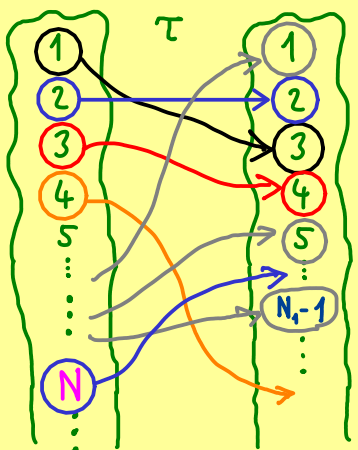
$$\sum_{k=1}^{\infty} a_{\tau(k)} \text{ is also abs. convergent and } \sum_{k=1}^{\infty} a_{\tau(k)} = \sum_{k=1}^{\infty} a_k.$$

Proof: Let  $\epsilon > 0$ . Cauchy criterion  $\Rightarrow \exists N_1 \in \mathbb{N} \forall n \geq m \geq N_1: \sum_{k=m}^n |a_k| < \epsilon$

$$\left| \sum_{k=1}^n a_k - \sum_{k=1}^n a_{\tau(k)} \right| = \left| A - \sum_{k=1}^{N_1-1} a_k + \sum_{k=1}^{N_1-1} a_k - \sum_{k=1}^n a_{\tau(k)} \right|$$

$$\leq \underbrace{\left| A - \sum_{k=1}^{N_1-1} a_k \right|}_{= \left| \sum_{k=N_1}^{\infty} a_k \right| = \lim_{n \rightarrow \infty} \left| \sum_{k=N_1}^n a_k \right| = (*)} + \underbrace{\left| \sum_{k=1}^{N_1-1} a_k - \sum_{k=1}^n a_{\tau(k)} \right|}_{< 2 \cdot \epsilon'} < 2 \cdot \epsilon'$$

$$\leq \lim_{n \rightarrow \infty} \sum_{k=N_1}^n |a_k| < \epsilon$$



For  $n \geq N$ :  $\{\tau(1), \tau(2), \dots, \tau(n)\} \supseteq \{1, 2, \dots, N_1-1\}$

$$\Rightarrow (*) = \left| \sum_{k=1}^n a_{\tau(k)} \right| \leq \sum_{k=1}^n |a_{\tau(k)}| \leq \sum_{j=N_1}^{\infty} |a_j| < \epsilon$$

$$\Rightarrow \forall \epsilon' > 0 \exists N \in \mathbb{N} \forall n \geq N: \left| \sum_{k=1}^n a_k - \sum_{k=1}^n a_{\tau(k)} \right| < \epsilon'$$