The Bright Side of Mathematics

Real Analysis – Part 16

Series: $\sum_{k=1}^{\infty} a_k$ is the sequence of partial sums $\sum_{k=1}^{n} a_k$

Example: geometric series $\sum_{k=0}^{\infty} q^k$, $q \in \mathbb{R}$

We show: $\sum_{k=0}^{\infty} q^k$ convergent $\iff$ $|q| < 1$

Question: $s_n = \sum_{k=0}^{n} q^k = ?$

For $q \neq 1$: $(1-q)\cdot\sum_{k=0}^{n} q^k = \sum_{k=0}^{n} q^k - \sum_{k=0}^{n} q^{k+1} = \sum_{k=0}^{n} q^k - \sum_{k=1}^{n+1} q^k = 1 - q^{n+1}$

$$s_n = \sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q}$$

$(s_n)_{n\in\mathbb{N}}$ convergent $\iff$ $(q^n)_{n\in\mathbb{N}}$ convergent to 0 $\iff$ $|q|<1$

For $|q|<1$: $\boxed{\sum_{k=0}^{\infty} q^k = \lim_{n\to\infty} s_n = \frac{1}{1-q}}$ geometric series

Example: Harmonic series

$$\sum_{k=1}^{\infty} \frac{1}{k} = \frac{1}{1} + \underline{\frac{1}{2} + \frac{1}{3} + \frac{1}{4}} + \cdots = \infty \quad \text{(divergent to infinity)}$$

Proof: $s_n = \sum_{k=1}^{n} \frac{1}{k}$ (sequence is monotonically increasing)

Show that $(s_n)_{n\in\mathbb{N}}$ is not bounded from above.

$$s_{2^m} = s_1 + (s_2 - s_1) + (s_4 - s_2) + (s_8 - s_4) + \cdots + (s_{2^m} - s_{2^{m-1}})$$

$$= s_1 + \sum_{j=1}^{m} (s_{2^j} - s_{2^{j-1}}) > s_1 + m\cdot\frac{1}{2} \xrightarrow{m\to\infty} \infty$$

because:

$$s_{2^j} - s_{2^{j-1}} = \sum_{k=2^{j-1}+1}^{2^j} \frac{1}{k} > \sum_{k=2^{j-1}+1}^{2^j} \frac{1}{2^j} = 2^{j-1}\cdot\frac{1}{2^j} = \frac{1}{2}$$