BECOME A MEMBER

ON STEADY

The Bright Side of
Mathematics

Real Analysis – Part 12

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers.

$$\Longrightarrow \quad \limsup_{n\to\infty} a_n \,,\ \liminf_{n\to\infty} a_n \in \mathbb{R} \cup \{\pm\infty\} = [-\infty, \infty]$$

Example: $(a_n)_{n\in\mathbb{N}} = \left((-1)^n \cdot n \right)_{n\in\mathbb{N}} = (-1, 2, -3, 4, -5, \dots)$

$$\limsup_{n\to\infty} a_n = \infty$$

$$\liminf_{n\to\infty} a_n = -\infty$$

Properties: (a) $(a_n)_{n\in\mathbb{N}}$ is convergent $\iff \limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n \notin \{\pm\infty\}$

(b) $(a_n)_{n\in\mathbb{N}}$ is divergent to ∞ $\iff \left(\limsup_{n\to\infty} a_n = \right) \liminf_{n\to\infty} a_n = \infty$

(c) $(a_n)_{n\in\mathbb{N}}$ is divergent to $-\infty$ $\iff \left(\liminf_{n\to\infty} a_n = \right) \limsup_{n\to\infty} a_n = -\infty$

(d) For $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$, we have:

$$\limsup_{n\to\infty} (a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

If $a_n, b_n \geq 0$: $$\limsup_{n\to\infty} (a_n \cdot b_n) \leq \limsup_{n\to\infty} a_n \cdot \limsup_{n\to\infty} b_n$$

(only if the right-hand side is defined)
$\infty - \infty$ not defined
$0 \cdot \infty$ not defined

$$\liminf_{n\to\infty} (a_n + b_n) \geq \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n$$

If $a_n, b_n \geq 0$: $$\liminf_{n\to\infty} (a_n \cdot b_n) \geq \liminf_{n\to\infty} a_n \cdot \liminf_{n\to\infty} b_n$$

Example: $(a_n)_{n\in\mathbb{N}} = (1, -1, 1, -1, 1, -1, 1, -1, \dots)$

$(b_n)_{n\in\mathbb{N}} = (0, 2, 0, 2, 0, 2, 0, 2, \dots)$

$(a_n + b_n)_{n\in\mathbb{N}} = (1, 1, 1, 1, 1, 1, 1, 1, \dots)$

$$1 = \limsup_{n\to\infty} (a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n = 1 + 2 = 3$$

$$1 = \liminf_{n\to\infty} (a_n + b_n) \geq \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n = -1 + 0 = -1$$

Example: $(a_n)_{n\in\mathbb{N}} = (1, 0, 1, 0, 1, 0, 1, 0, \dots)$

$(b_n)_{n\in\mathbb{N}} = (0, 2, 0, 2, 0, 2, 0, 2, \dots)$

$(a_n \cdot b_n)_{n\in\mathbb{N}} = (0, 0, 0, 0, 0, 0, 0, 0, \dots)$

$$0 = \limsup_{n\to\infty} (a_n \cdot b_n) \leq \limsup_{n\to\infty} a_n \cdot \limsup_{n\to\infty} b_n = 1 \cdot 2 = 2$$