Bolzano-Weierstrass Theorem

Next we present the famous Theorem of Bolzano-Weierstraß.

Theorem 1. Theorem of Bolzano-Weierstraß

Let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathbb{R}$. Then there exists some convergent subsequence $(a_{n_k})_{k\in\mathbb{N}}$.

Proof: Since $(a_n)_{n\in\mathbb{N}}$ is bounded, there exist some $A, B \in \mathbb{R}$ such that for all $n \in \mathbb{N}$ holds $A \leq a_n \leq B$. We will now successively construct subintervals $[A_n, B_n] \subset [A, B]$ which still include infinitely many sequence elements of $(a_n)_{n\in\mathbb{N}}$.

Inductively define $A_0 = A$, $B_0 = B$ and for $k \geq 1$,

a) $A_k = A_{k-1}$, $B_k = \frac{A_{k-1}+B_{k-1}}{2}$, if the interval $[A_{k-1}, \frac{A_{k-1}+B_{k-1}}{2}]$ contains infinitely many sequence elements of $(a_n)_{n\in\mathbb{N}}$, and

b) $A_k = \frac{A_{k-1}+B_{k-1}}{2}$, $B_k = B_{k-1}$, else.

By the construction of A_k and B_k, we have that each interval $[A_k, B_k]$ has infinitely many sequence elements of $(a_n)_{n\in\mathbb{N}}$. We furthermore have $B_1 - A_1 = \frac{1}{2}(B - A)$, $B_2 - A_2 = \frac{1}{4}(B - A)$, $\ldots$, $B_k - A_k = \frac{1}{2^k}(B - A)$. Moreover, the sequence $(A_n)_{n\in\mathbb{N}}$ is monotonically increasing and bounded from above by B, i.e., it is convergent by the Theorem about bounded monotonic sequences. The relation $B_k - A_k = \frac{1}{2^k}(B - A)$ moreover implies that $(B_n)_{n\in\mathbb{N}}$ is also convergent and has the same limit as $(A_n)_{n\in\mathbb{N}}$. Denote

$$a = \lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n.$$

Define a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ by $n_1 = 1$ and n_k with $n_k > n_{k-1}$ and $a_{n_k} \in [A_k, B_k]$ (which is possible since $[A_k, B_k]$ contains infinitely many elements of $(a_n)_{n\in\mathbb{N}}$). Then $A_k \leq a_{n_k} \leq B_k$. The Theorem about monotonic sequences then implies that

$$a = \lim_{k\to\infty} a_{n_k}.$$

□

Exercise 2.

A sequence (a_n) is called periodic *if there is $p \in \mathbb{N}$ such that $\forall n \in \mathbb{N} : a_n = a_{n+p}$. The smallest p such that the above equation holds is called period of the sequence (a_n). For $\alpha \in \mathbb{R}$ consider the sequence given by $a_n := \sin(\alpha\pi n)$.*

1. *Prove that (a_n) is periodic if and only if $\alpha \in \mathbb{Q}$.*
   Hint: $\sin(x) = \sin(y) \neq 0 \Leftrightarrow x = y + 2k\pi$ *and* $\sin(x) = 0 \Leftrightarrow x = k\pi$ *for some* $k \in \mathbb{Z}$
2. *For $\alpha = \frac{2}{3}$ find a converging subsequence.*
3. *For $\alpha = \frac{4}{5}$ determine the period of (a_n).*
4. *For $\alpha = \frac{1}{2}$ find all limit points of (a_n).*

Exercise 3.

We consider the metric space $M := \mathbb{R}^2$ *equipped with a metric* d *defined for all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ *by* $d(\mathbf{x}, \mathbf{y}) := \|\mathbf{x} - \mathbf{y}\|$ *where (*$\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$*)* $\|\mathbf{x}\| := \sqrt{x_1^2 + x_2^2}$ *is the standard euclidean metric. Prove the Theorem of Bolzano-Weierstraß for bounded sequences* $(\mathbf{x}_n)$ *in* $\mathbb{R}^2$*. So if* $(\mathbf{x}_n)$ *in* $\mathbb{R}^2$ *is a bounded sequence then there is a convergent subsequence* $(\mathbf{x}_{n_k})$*.*