Theorem on Limits

In the following, we present some results that allow us to determine some further limits.

Theorem 1. Formulae for convergent sequences

Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be a convergent sequences in $\mathbb{F}$. Then the following holds true:

(i) $(a_n + b_n)_{n\in\mathbb{N}}$ is convergent with

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n.$$

(ii) $(a_n \cdot b_n)_{n\in\mathbb{N}}$ is convergent with

$$\lim_{n\to\infty}(a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n.$$

(iii) If $\lim_{n\to\infty} b_n \neq 0$ and $b_n \neq 0$ for all $n \in \mathbb{N}$, then the sequence $(\frac{a_n}{b_n})_{n\in\mathbb{N}}$ is convergent with

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}.$$

Proof. Let $a = \lim\limits_{n\to\infty} a_n$ and $b = \lim\limits_{n\to\infty} b_n$.

(i) Let $\varepsilon > 0$ be arbitrary. Then
there exists some N_1 such that for all $n \geq N_1$ holds $|a - a_n| < \frac{\varepsilon}{2}$, and
there exists some N_2 such that for all $n \geq N_2$ holds $|b - b_n| < \frac{\varepsilon}{2}$.
Now choose $N = \max\{N_1, N_2\}$. Then for all $n \geq N$ holds

$$\begin{aligned} &|(a+b) - (a_n + b_n)| \\ &= |(a - a_n) + (b - b_n)| \leq |a - a_n| + |b - b_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

(ii) Due to the theorem from the last video, both sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are bounded. Choose numbers $c_1, c_2 > 0$ such that $|a_n| < c_1$ and $|b_n| < c_2$ for all $n \in \mathbb{N}$. Define

$$c = \max\{c_1, c_2\}.$$

Let $\varepsilon > 0$. Convergence of $(a_n)_{n\in\mathbb{N}}$ to a implies that there exists some N_1 such that $|a - a_n| < \frac{\varepsilon}{2c}$ for all $n \geq N_1$. Furthermore, the convergence of $(b_n)_{n\in\mathbb{N}}$ to b implies that there exists some N_2 such that $|b - b_n| < \frac{\varepsilon}{2c}$ for all $n \geq N_2$. Now define $N = \max\{N_1, N_2\}$. Then for $n \geq N$ holds

$$\begin{aligned} &|ab - a_n b_n| \\ &= |(ab - a_n b) + (a_n b - a_n b_n)| \leq |ab - a_n b| + |a_n b - a_n b_n| \\ &= |a - a_n| \cdot |b| + |a_n| \cdot |b - b_n| \leq |a - a_n| \cdot c + c \cdot |b - b_n| \\ &< \frac{\varepsilon}{2c} \cdot c + c \cdot \frac{\varepsilon}{2c} = \varepsilon. \end{aligned}$$

(iii) First we specialize to the case where $(a_n)_{n\in\mathbb{N}}$ is the constant sequence $(a_n)_{n\in\mathbb{N}} = (1,1,1,1,\ldots)$. Due to $b > 0$, we have the existence of some N_1 such that for all $n \geq N_1$ holds

$$|b_n - b| < \frac{|b|}{2}.$$

This just follows by an application of the "ε-criterion" to $\varepsilon = \frac{|b|}{2}$. In particular, this leads to $|b| \leq |b_n - b| + |b_n| < \frac{|b|}{2} + |b_n|$ and thus $|b_n| > \frac{|b|}{2}$.
Let $\varepsilon > 0$. Let N_2 such that for all $n \geq N_2$ holds

$$|b_n - b| < \frac{\varepsilon \cdot |b|^2}{2}.$$

Then for $n \geq N := \max\{N_1, N_2\}$ holds

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \frac{1}{|b_n| \cdot |b|} \cdot |b_n - b| < \frac{2}{|b|^2} \cdot |b_n - b| < \frac{2}{|b|^2} \cdot \frac{\varepsilon \cdot |b|^2}{2} = \varepsilon.$$

So far, we have shown that for some sequence $(b_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} b_n \neq 0$ and $b_n \neq 0$ for all $n \in \mathbb{N}$ holds

$$\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{\lim\limits_{n\to\infty} b_n}.$$

The general statement for the sequence $(\frac{a_n}{b_n})_{n\in\mathbb{N}}$ follows by rewriting

$$\left(\frac{a_n}{b_n}\right)_{n\in\mathbb{N}} = \left(a_n \cdot \frac{1}{b_n}\right)_{n\in\mathbb{N}}$$

and applying the multiplication rule (ii).

□

Remark 2.

(a) Since the constant sequence $(a)_{n\in\mathbb{N}} = (a,a,a,\ldots)$ is, of course, convergent to a, statement (ii) also implies the formula

$$\lim_{n\to\infty} (a \cdot b_n) = a \cdot \lim_{n\to\infty} b_n.$$

(b) For $k \in \mathbb{N}$, a k-times application of statement (ii) yields that for some convergent sequence $(a_n)_{n\in\mathbb{N}}$, also the sequence $(a_n^k)_{n\in\mathbb{N}}$ is convergent with

$$\lim_{n\to\infty} a_n^k = \left(\lim_{n\to\infty} a_n\right)^k.$$

Exercise 3.

The following sequences are convergent. Find their limits and justify your steps by clearly mentioning the properties/previously known results that you have used.

(a) $a_n := 3 - \frac{12}{n^3}$

(b) $b_n := \frac{5n^4 - 7n^3 + 12n}{2n^4 + 8n - 4}$