Bounded Sequences

Definition 1. Boundedness of sequences

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. Then $(a_n)_{n\in\mathbb{N}}$ is called

(a) bounded if there exists some $c \in \mathbb{R}$ such that for all $n \in \mathbb{N}$ holds $|a_n| \leq c$;

(b) unbounded if it is not bounded, i.e., for all $c \in \mathbb{R}$, there exists some $n \in \mathbb{N}$ with $|a_n| > c$.

We can make the definition also more exact if we deal with real numbers:

Definition 2. Bounded from above and below

*A **real** sequence $(a_n)_{n\in\mathbb{N}}$ is called*

(a) bounded from above if there exists some $c \in \mathbb{R}$ with $a_n \leq c$ for all $n \in \mathbb{N}$.

(b) bounded from below if there exists some $c \in \mathbb{R}$ with $a_n \geq c$ for all $n \in \mathbb{N}$.

Theorem 3.

Let $(a_n)_{n\in\mathbb{N}}$ be a convergent sequence in $\mathbb{R}$. Then $(a_n)_{n\in\mathbb{N}}$ is bounded.

Proof. Suppose that $\lim_{n\to\infty} a_n = a$. Take $\varepsilon = 1$. Then there exists some N such that for all $n \geq N$ holds $|a_n - a| < 1$. Thus, for all $n \geq N$ holds

$$|a_n| = |a_n - a + a| \leq |a_n - a| + |a| < 1 + |a|.$$

Now choose

$$c = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, |a| + 1\}$$

and consider some arbitrary sequence element a_k.
If $k < N$, then $|a_k| \leq \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|\} \leq c$.
In the case $k \geq N$, the above calculations lead to $|a_k| < |a| + 1 \leq c$.
Altogether, this implies that $|a_k| \leq c$ for all $k \in \mathbb{N}$, so $(a_n)_{n\in\mathbb{N}}$ is bounded by c. □

Remark 4.

For a convergent sequence $(a_n)_{n\in\mathbb{N}}$ that is bounded by c, we can also deduce from the above argumentation that for the limit a holds $|a| \leq c$:
Suppose that $\lim_{n\to\infty} a_n = a \in \mathbb{R}$, then for an arbitrary $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|a - a_n| < \varepsilon$ for all $n \geq N$. Hence $|a| = |a - a_n + a_n| \leq |a - a_n| + |a_n| \leq \varepsilon + c$.
Since $\varepsilon > 0$ can be chosen arbitrarily small this implies $|a| \leq c$.

Theorem 5. Uniqueness of the limit of a convergent sequence

*Let $(a_n)_{n\in\mathbb{N}}$ be a convergent sequence in $\mathbb{R}$. Then there exists only **one** limit.*

Proof by contradiction: Let $a, b \in \mathbb{R}$ be two distinct ($a \neq b$) limits of $(a_n)_{n\in\mathbb{N}}$. Then we have $|a - b| > 0$ and for $\varepsilon = \frac{1}{4}|a - b|$ the following statements are fulfilled:

There exists some N_1 such that for all $n \geq N_1$ holds $|a_n - a| < \varepsilon$.
There exists some N_2 such that for all $n \geq N_2$ holds $|a_n - b| < \varepsilon$.
Let $n \geq \max\{N_1, N_2\}$. Then

$$\begin{aligned} |a - b| &= |a - a_n + a_n - b| \\ &\leq |a - a_n| + |a_n - b| \leq \varepsilon + \varepsilon = 2\varepsilon = \frac{1}{2} \cdot |a - b| \end{aligned}$$

Thus, $|a - b| \leq \frac{1}{2}|a - b|$. However, this implies $|a - b| \leq 0$ which is only fulfilled, if $|a - b| = 0$, or, equivalently $a = b$. This is a contradiction to the initial assumption. □

Exercise 6.

Are the following sequences bounded, bounded from below, bounded from above or unbounded?

(a)

$$a_n = \frac{(10n-1)!}{(10+1)!}$$

(b)

$$a_n = (-1)^n$$

(c)

$$a_n = n(-1)^n$$

(d)

$$a_n = \frac{n}{n+1}$$

(e)

$$a_n = \begin{cases} 2^n & \text{for } n \text{ even} \\ 3 & \text{for } n \text{ odd} \end{cases}$$

Exercise 7.

Give examples or counterexamples for the given statements.

(a) *If $(a_n)_{n\in\mathbb{N}}$ is convergent, then also $(|a_n|)_{n\in\mathbb{N}}$ is convergent.*

(b) *If $(|a_n|)_{n\in\mathbb{N}}$ is convergent, then also $(a_n)_{n\in\mathbb{N}}$ is convergent.*

(c) *If $(a_n)_{n\in\mathbb{N}}$ and $(a_n + b_n)_{n\in\mathbb{N}}$ converge, then $(b_n)_{n\in\mathbb{N}}$ does not converge.*

(d) *There are sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ that don't converge but $(a_n + b_n)_{n\in\mathbb{N}}$ does converge.*