Convergence of Sequences

Definition 1.

Let M be a set. A sequence in M is a map $a : \mathbb{N} \to M$ or $a : \mathbb{N}_0 \to M$.

We use the following symbols for sequences:

$$(a_n)_{n\in\mathbb{N}}, \qquad (a_n), \qquad (a_n)_{n=1}^{\infty}, \quad (a_1, a_2, a_3, \dots).$$

Remark 2.

For our course here, M is usually a real subset ($M \subset \mathbb{R}$), but later M can also be a complex subset ($M \subset \mathbb{C}$).

Example 3. (a) $a_n = (-1)^n$, then $(a_n)_{n\in\mathbb{N}} = ((-1)^n)_{n\in\mathbb{N}} = (-1, 1, -1, 1, -1, 1, \dots)$
(b) $a_n = \frac{1}{n}$, then $(a_n)_{n\in\mathbb{N}} = (\frac{1}{n})_{n\in\mathbb{N}} = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \dots)$
(c) $a_n = \frac{1}{2^n}$, then $(a_n)_{n\in\mathbb{N}} = (\frac{1}{2^n})_{n\in\mathbb{N}} = (1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \dots)$

Next we define the notions of convergence and limits:

Definition 4. Convergence/divergence of sequences

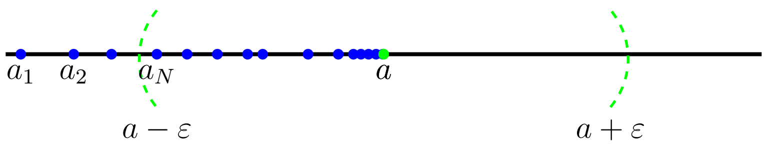
Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. We say that

- *$(a_n)_{n\in\mathbb{N}}$ is convergent to $a \in \mathbb{R}$ if for all $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that for all $n \geq N$ holds $|a_n - a| < \varepsilon$. In this case, we write*

$$\lim_{n\to\infty} a_n = a.$$

- *$(a_n)_{n\in\mathbb{N}}$ is divergent if it is not convergent, i.e., for all $a \in \mathbb{R}$ holds: There exists some $\varepsilon > 0$ such that for all N there exists some $n > N$ with $|a_n - a| \geq \varepsilon$.*

Convergence for real sequences means that if you give any small distance ε, one finds that all sequence members a_n lie in the interval $(a - \varepsilon, a + \varepsilon)$ with the exception of only *finitely* many.



The next exercises are explained in detail. The first one is covered in the video and the next one is just very similar.

Exercise 5.

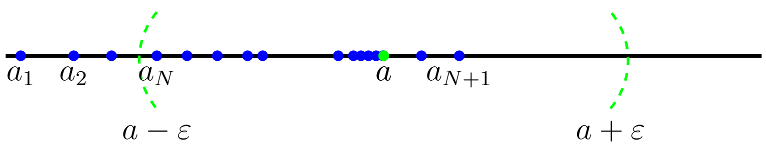
- *Show:* $(a_n)_{n\in\mathbb{N}}$ *with* $a_n = (1/n)$ *is convergent with limit* 0.
- *Show:* $(b_n)_{n\in\mathbb{N}}$ *with* $b_n = (1/\sqrt{n})$ *is convergent with limit* 0.

Proof of (b). Let $\varepsilon > 0$. Choose $N > \frac{1}{\varepsilon^2}$. Then for all $n \geq N$, we have

$$|b_n - 0| = \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{N}} < \varepsilon$$

This means b_n is arbitrarily close to 0, eventually. □

Always keep the picture in mind:



Convergence means: Outside any ε-neighbourhood of a only finitely many elements of the sequence exist.

We will need the following inequality for the next convergence proof such that you should check that the inequality is correct.

Exercise 6. Bernoulli's inequality

Prove the following inequality by induction: $\forall n \in \mathbb{N},\ h \geq -1:\ (1+h)^n \geq 1 + hn$.

Now, we present an important example. Try to solve it first for yourself and then compare your solution to the solution here.

Exercise 7.

For $q \in \mathbb{R}\setminus\{0\}$ *with* $|q| < 1$, *the sequence* $(q^n)_{n\in\mathbb{N}}$ *converges to 0.*

Proof. $|q| < 1$ gives rise to $\frac{1}{|q|} > 1$, and therefore $\frac{1}{|q|} - 1 > 0$. Hence, we are able to apply Bernoulli's inequality (see above) in the following way:

$$\frac{1}{|q|^n} = \left(1 + \left(\frac{1}{|q|} - 1\right)\right)^n = \left(1 + \left(\frac{1-|q|}{|q|}\right)\right)^n \geq 1 + n \cdot \left(\frac{1-|q|}{|q|}\right),$$

and thus

$$|q|^n \leq \frac{1}{1 + n \cdot \left(\frac{1-|q|}{|q|}\right)} = \frac{|q|}{|q| + n \cdot (1-|q|)}.$$

Now let $\varepsilon > 0$ (be arbitrary):

Choose

$$N > \frac{|q|}{\varepsilon \cdot (1-|q|)} - \frac{|q|}{1-|q|} + 1\,.$$

Then for all $n \geq N$ holds

$$n > \frac{|q|}{\varepsilon \cdot (1 - |q|)} - \frac{|q|}{1 - |q|}$$

and thus

$$n \cdot (1 - |q|) > \frac{|q|}{\varepsilon} - |q|.$$

This leads to

$$|q| + n \cdot (1 - |q|) > \frac{|q|}{\varepsilon},$$

and

$$\frac{|q|}{|q| + n \cdot (1 - |q|)} < \varepsilon.$$

The above calculations now imply

$$|q^n - 0| = |q|^n \leq \frac{|q|}{|q| + n \cdot (1 - |q|)} < \varepsilon\,,$$

which closes the proof. □