BECOME A MEMBER

ON STEADY

The Bright Side of Mathematics

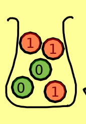
Probability Theory - Part 31

Assumptions of the central limit theorem: $(X_k)_{k \in \mathbb{N}}$ i.i.d. with $\mathrm{Var}(X_1) < \infty$.

part 28

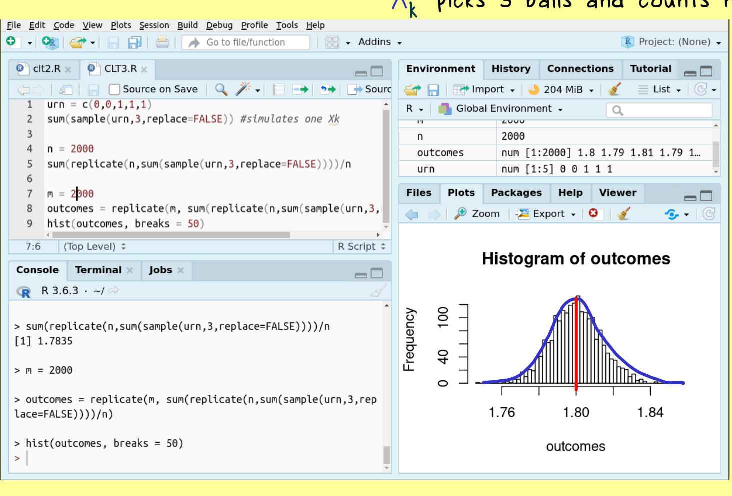
$$\Longrightarrow \quad \overline{X}_n := \frac{1}{n}\sum_{k=1}^{n} X_k \quad \text{satisfies} \quad \mathbb{E}(\overline{X}_n) = \mathbb{E}(X_1), \quad \mathrm{Var}(\overline{X}_n) = \frac{\mathrm{Var}(X_1)}{n}$$

Example:



urn model without replacement = hypergeometric distribution (part 6)

X_k picks 3 balls and counts numbers of ①



$$\mathbb{E}(X_1) = \frac{9}{5} = 1.8$$

$$\overline{X}_n := \frac{1}{n}\sum_{k=1}^{n} X_k$$

What is the distribution?

close to normal distribution!

$$\mathrm{Var}(\overline{X}_n) = \frac{\mathrm{Var}(X_1)}{n}$$

Standardize the random variable: $\mu := \mathbb{E}(X_1)$, $\sigma := \sqrt{\mathrm{Var}(X_1)}$

(1) expectation should be zero: $\overline{X}_n - \mu$

(2) variance should be one: $(\overline{X}_n - \mu) / \left(\frac{\sigma}{\sqrt{n}}\right)$

Central limit theorem: For $(X_k)_{k \in \mathbb{N}}$ i.i.d. with $\mathrm{Var}(X_1) < \infty$, define:

$$Y_n := \left(\frac{1}{n}\sum_{k=1}^{n} X_k - \mu\right) \cdot \left(\frac{\sigma}{\sqrt{n}}\right)^{-1} \quad \text{where } \mu := \mathbb{E}(X_1), \quad \sigma := \sqrt{\mathrm{Var}(X_1)}$$

Then the cdf of Y_n converges to the cdf of Normal$(0, 1^2)$:

$$\mathbb{P}(Y_n \leq x) \xrightarrow{n \to \infty} \Phi(x) \quad \text{for every } x \in \mathbb{R}$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2} \, dt$$