The Bright Side of Mathematics

Probability Theory - Part 28

(theoretical) probability of event $\mathbb{P}(A)$ $\longleftrightarrow$ (law of large numbers) empirical probability of event $\frac{\text{number of outcomes in } A}{\text{total number}}$ = relative frequency of the event

Example: coin toss: $\Omega_0 = \{H, T\}$, $\mathbb{P}_0(\{H\}) = \mathbb{P}_0(\{T\}) = \frac{1}{2}$

repeat random experiment: $\Omega = \Omega_0 \times \Omega_0 \times \cdots$

$\mathbb{P}$ = product measure

define random variables: $X_k : \Omega \to \mathbb{R}$, $X_k(\omega) = \begin{cases} 1, & \omega_k = H \\ 0, & \omega_k = T \end{cases}$ (H in kth toss)

let's look at n tosses: $\overline{X}_n := \frac{1}{n}\sum_{k=1}^{n} X_k : \Omega \to \mathbb{R}$

(relative frequency of heads in the first n tosses)

we expect: $\overline{X}_n \xrightarrow{n \to \infty} \frac{1}{2}$ What does this convergence mean?

Weak law of large numbers: $X_k : \Omega \to \mathbb{R}$ random variables.

Let $(X_k)_{k \in \mathbb{N}}$ be independent and identically distributed (= i.i.d.)

$\left[\mathbb{P}\left((X_j \leq x_j)_{j \in J}\right) = \prod_{j \in J} \mathbb{P}(X_j \leq x_j) \text{ for all } x_j \in \mathbb{R} \text{ for all finite } \emptyset \neq J \subseteq \mathbb{N}\right]$ $\left[\mathbb{P}_{X_k}(B) = \mathbb{P}_{X_1}(B) \text{ for all } k \in \mathbb{N} \text{ for all Borel sets } B \subseteq \mathbb{R}\right]$

and $\mathbb{E}(|X_1|) < \infty$.

Then for $\mu := \mathbb{E}(X_1)$ and for all $\varepsilon > 0$:

$$\mathbb{P}\left(\left|\frac{1}{n}\sum_{k=1}^{n} X_k - \mu\right| \geq \varepsilon\right) \xrightarrow{n \to \infty} 0$$

We say $\overline{X}_n := \frac{1}{n}\sum_{k=1}^{n} X_k$ converges in probability to the expected value μ.

Proof: for the case: $\mathrm{Var}(X_1) < \infty$ ($= \sigma^2$)

We have: $\mathbb{E}(\overline{X}_n) = \mathbb{E}\left(\frac{1}{n}\sum_{k=1}^{n} X_k\right) = \frac{1}{n}\sum_{k=1}^{n} \mathbb{E}(X_k) = \mu$

$\mathrm{Var}(\overline{X}_n) = \mathrm{Var}\left(\frac{1}{n}\sum_{k=1}^{n} X_k\right) = \frac{1}{n^2}\sum_{k=1}^{n} \mathrm{Var}(X_k) = \frac{\sigma^2}{n}$

By Chebyshev's inequality:

$$\mathbb{P}\left(|\overline{X}_n - \underbrace{\mathbb{E}(\overline{X}_n)}_{\mu}| \geq \varepsilon\right) \leq \frac{\mathrm{Var}(\overline{X}_n)}{\varepsilon^2} \text{ for any } \varepsilon > 0.$$

$$= \frac{\sigma^2}{\varepsilon^2} \cdot \frac{1}{n} \xrightarrow{n \to \infty} 0 \qquad \square$$