Probability Theory - Part 26

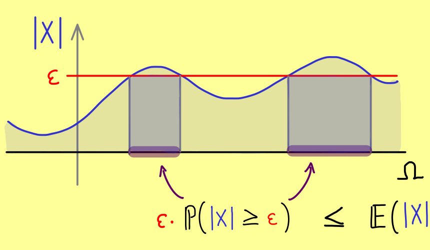
$(\Omega, \mathcal{A}, \mathbb{P})$ probability space

Markov's inequality: $X : \Omega \longrightarrow \mathbb{R}$ random variable.

Then $|X| : \Omega \longrightarrow [0, \infty)$ satisfies:

$$\mathbb{P}(|X| \geq \varepsilon) \leq \frac{\mathbb{E}(|X|^p)}{\varepsilon^p} \quad \text{for any } \varepsilon > 0 , \; p > 0$$

picture for $p = 1$:



$$\varepsilon \cdot \mathbb{P}(|X| \geq \varepsilon) \leq \mathbb{E}(|X|)$$

Proof:

We have: $|X(\omega)| \geq \varepsilon \iff |X(\omega)|^p \geq \varepsilon^p$

indicator function

And: $\varepsilon^p \, \mathbb{P}(|X| \geq \varepsilon) = \varepsilon^p \cdot \mathbb{P}(|X|^p \geq \varepsilon^p) = \varepsilon^p \cdot \mathbb{E}(\mathbb{1}_{\{|X|^p \geq \varepsilon^p\}})$

$$= \mathbb{E}(\varepsilon^p \cdot \mathbb{1}_{\{|X|^p \geq \varepsilon^p\}}) \leq \mathbb{E}(|X|^p) \qquad \square$$

Chebyshev's inequality: $X : \Omega \longrightarrow \mathbb{R}$ random variable where $\mathbb{E}(|X|) < \infty$.

Then:

$$\mathbb{P}\big(|X - \mathbb{E}(X)| \geq \varepsilon \big) \leq \frac{\mathrm{Var}(X)}{\varepsilon^2} \quad \text{for any } \varepsilon > 0.$$

Proof: Define: $\tilde{X} := X - \mathbb{E}(X)$. Hence: $\mathrm{Var}(X) = \mathrm{Var}(\tilde{X}) = \mathbb{E}(\tilde{X}^2)$

$$\mathbb{P}(|X - \mathbb{E}(X)| \geq \varepsilon) = \mathbb{P}(|\tilde{X}| \geq \varepsilon) \leq \frac{\mathbb{E}(|\tilde{X}|^2)}{\varepsilon^2} = \frac{\mathrm{Var}(X)}{\varepsilon^2}$$

Markov's inequality for $p = 2$ $\qquad \square$