Probability Theory – Part 24

Definition: Let $(X_t)_{t \in T}$ be a stochastic process with $T \subseteq \mathbb{Z}$ (discrete-time) or $T \subseteq \mathbb{R}$ (continuous-time).

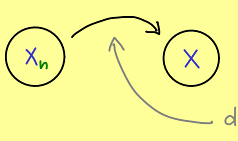
We call $(X_t)_{t \in T}$ Markov process or Markov chain if

for all $n \in \mathbb{N}$, $t_1, t_2, \dots, t_n, t \in T$, $t_1 < t_2 < \dots < t_n < t$,

and $x_1, x_2, \dots, x_n, x \in \mathbb{R}$, we have:

$$\mathbb{P}(X_t = x \mid X_{t_1} = x_1, X_{t_2} = x_2, \dots, X_{t_n} = x_n)$$
$$= \mathbb{P}(X_t = x \mid X_{t_n} = x_n)$$

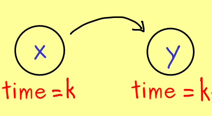
for discrete-time Markov chain:



depends only on X_n, X, t_n

$$p_{x,y}(k, k+1) = \mathbb{P}(X_{k+1} = y \mid X_k = x)$$

transition probability from x to y at time k



If $p_{x,y}(k, k+1)$ does not depend on k, then we say:

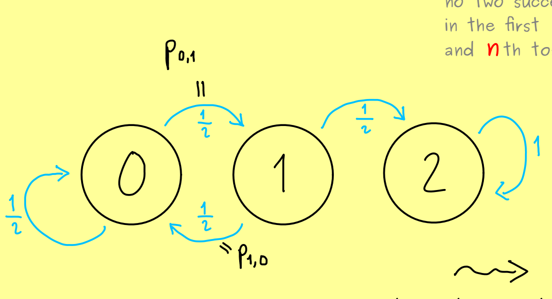
the Markov chain is time-homogeneous

Example: toss a coin again and again until two successive heads occur

$$X_n : \{H, T\}^{\mathbb{N}} \longrightarrow \{0, 1, 2\}$$

$n \in \mathbb{N}$ discrete time

- 0: no two successive heads in the first n tosses and nth toss is "tails"
- 1: no two successive heads in the first n tosses and nth toss is "heads"
- 2: two successive heads in the first n tosses



$\rightsquigarrow$ transition matrix

$$P = \begin{pmatrix} p_{0,0} & p_{0,1} & p_{0,2} \\ p_{1,0} & p_{1,1} & p_{1,2} \\ p_{2,0} & p_{2,1} & p_{2,2} \end{pmatrix}$$

Here:
$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

Start the game with $q^0 = (1, 0, 0)$ $\xrightarrow{\text{one time-step}}$ $q^1 = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$

$\xrightarrow{\text{one time-step}}$ $q^2 = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$

$$q^2 = q^1 P \quad \text{(vector-matrix-multiplication)}$$

Law of total probability

$$\rightsquigarrow \quad q^n = q^0 P^n$$

$\xrightarrow{n \to \infty}$ $(0, 0, 1)$?