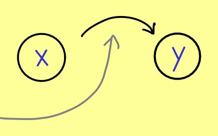
The Bright Side of Mathematics

Probability Theory - Part 25

stochastic process: $(X_t)_{t \in T}$ ← subset of $\mathbb{Z}$ or $\mathbb{R}$

discrete-time Markov chains + time-homogeneous:

depends only on x and y

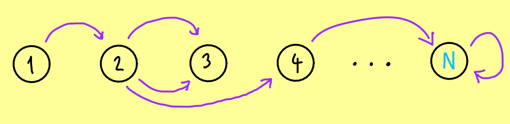


$$p_{x,y} := \mathbb{P}(X_{k+1} = y \mid X_k = x) \quad \text{independent of } k \in T \subseteq \mathbb{Z}$$

↪ transition matrix $\mathcal{P} = (p_{x,y})_{x,y}$

Important:
- entries of $\mathcal{P}$ lie in $[0,1]$
- $\mathcal{P}$ acts on row vectors from the right

General example: $X_k : \Omega \to \{1, 2, \dots, N\}$



start at $k=0$: probability mass function of X_0 (pmf of $\mathbb{P}_{X_0}$)

is given by a row vector $q^0 \in \mathbb{R}^{1 \times N}$

$$(q^0)_m = \mathbb{P}(X_0 = m)$$

at $k=1$:

$$(q^1)_m = \mathbb{P}(X_1 = m) = \sum_{i=1}^{N} \mathbb{P}(X_1 = m \mid B_i) \cdot \mathbb{P}(B_i)$$

↑ law of total probability $\bigcup_{i=1}^{N} B_i = \Omega$ (disjoint union!)

$B_i = \{X_0 = i\}$

$$= \sum_{i=1}^{N} \mathbb{P}(B_i) \cdot \mathbb{P}(X_1 = m \mid B_i)$$

$$= \sum_{i=1}^{N} \underbrace{\mathbb{P}(X_0 = i)}_{(q^0)_i} \cdot \underbrace{\mathbb{P}(X_1 = m \mid X_0 = i)}_{p_{i,m}} = (q^0 \mathcal{P})_m$$

by induction: $q^k = q^0 \cdot \mathcal{P}^k$

Definition: $q \in \mathbb{R}^{1 \times N}$ is called a stationary distribution for the Markov chain if

$$q\mathcal{P} = q \qquad \left(\text{and } q_m \in [0,1], \ \sum_m q_m = 1\right)$$

Note: $q\mathcal{P} = q \iff \mathcal{P}^T q^T = q^T \iff \mathcal{P}^T q^T = 1 \cdot q^T$

(q^T: column vector; 1: eigenvalue; q^T: eigenvector)

Example:

$$\mathcal{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \mathrm{Ker}(\mathcal{P}^T - 1 \cdot \mathbb{1}) = \mathrm{Ker}\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -1 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

row operations

$$= \mathrm{Ker}\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{Span}\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)$$

$\Longrightarrow$ only stationary distribution $q = (0, 0, 1)$