The Bright Side of
Mathematics

Probability Theory – Part 22

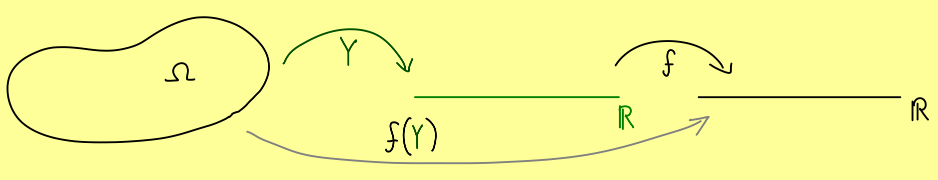
Recall: $X : \Omega \longrightarrow \mathbb{R}$ discrete, B event with $\mathbb{P}(B) > 0$

$$\mathbb{E}(X \mid B) = \int_\Omega X \, d\mathbb{P}(\cdot \mid B) = \sum_x x \cdot \mathbb{P}(X = x \mid B)$$

Consider $Y : \Omega \longrightarrow \mathbb{R}$ discrete, $B = \{Y = y\}$.

Define: $$f(y) := \mathbb{E}(X \mid Y = y) = \sum_x x \, \frac{\mathbb{P}(X = x \text{ and } Y = y)}{\mathbb{P}(Y = y)}$$

(joint pmf of X and Y)



$f(Y) : \Omega \longrightarrow \mathbb{R}$ is called the conditional expectation of X given Y and denoted by $\mathbb{E}(X \mid Y)$

Example: die throw, $\Omega = \{1, \dots, 6\}$, $X : \Omega \longrightarrow \mathbb{R}$ checks if number is even

$$X(\omega) = \begin{cases} 1, & \omega \in \{2,4,6\} \\ 0, & \text{else} \end{cases}$$

$Y : \Omega \longrightarrow \mathbb{R}$ checks if number is the highest

$$Y(\omega) = \begin{cases} 1, & \omega = 6 \\ 0, & \text{else} \end{cases}$$

$$\mathbb{E}(X \mid Y)(\omega) = \begin{cases} \mathbb{E}(X \mid Y = 0) = \sum_{x=0,1} x \, \frac{\mathbb{P}(X = x \text{ and } Y = 0)}{\mathbb{P}(Y = 0)} = \frac{\frac{2}{6}}{\frac{5}{6}} = \frac{2}{5}, & \omega \in \{1, \dots, 5\} \\ \mathbb{E}(X \mid Y = 1) = \sum_{x=0,1} x \, \frac{\mathbb{P}(X = x \text{ and } Y = 1)}{\mathbb{P}(Y = 1)} = \frac{\frac{1}{6}}{\frac{1}{6}} = 1, & \omega = 6 \end{cases}$$

Definition for (abs.) continuous case: $(X, Y) : \Omega \longrightarrow \mathbb{R}^2$ with pdf $f_{(X,Y)} : \mathbb{R}^2 \to \mathbb{R}$

$$g(y) := \mathbb{E}(X \mid Y = y) = \int_\mathbb{R} x \cdot \underbrace{\frac{f_{(X,Y)}(x,y)}{f_Y(y)}}_{\text{conditional density}} \, dx$$

$\mathbb{E}(X \mid Y) = g(Y) = g \circ Y$ is called the conditional expectation of X given Y

Properties: (a) X, Y independent $\Longrightarrow$ $\mathbb{E}(X \mid Y) = \mathbb{E}(X)$ and
$\mathbb{E}(X \cdot Y \mid Y) = \mathbb{E}(X) \cdot Y$

(b) $\mathbb{E}(X \mid X) = X$

(c) $\mathbb{E}(\mathbb{E}(X \mid Y)) = \mathbb{E}(X)$ (Law of total probability)