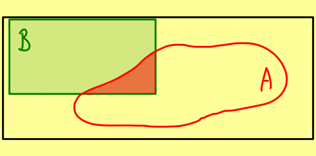
Probability Theory – Part 21

<u>conditional probability:</u>



$\mathbb{P}(\cdot \mid B) : A \mapsto \mathbb{P}(A \mid B)$

is probability measure $(\mathbb{P}(B) > 0)$

Definition: $(\Omega, \mathcal{A}, \mathbb{P})$ probability space, $B \in \mathcal{A}$ with $\mathbb{P}(B) > 0$

$(\Rightarrow (\Omega, \mathcal{A}, \mathbb{P}(\cdot \mid B))$ probability space$)$

For a random variable $X : \Omega \longrightarrow \mathbb{R}$, we define:

$$\mathbb{E}(X) = \int_\Omega X \, d\mathbb{P} \qquad \text{(expectation of } X\text{)}$$

$$\mathbb{E}(X \mid B) = \int_\Omega X \, d\mathbb{P}(\cdot \mid B) \qquad \text{(conditional expectation of } X \text{ given } B\text{)}$$

Remember:

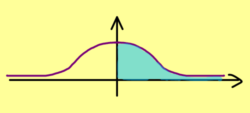
$$\mathbb{E}(X \mid B) = \frac{1}{\mathbb{P}(B)} \int_\Omega X \mathbb{1}_B \, d\mathbb{P} = \frac{1}{\mathbb{P}(B)} \mathbb{E}(\mathbb{1}_B X)$$

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{1}{\mathbb{P}(B)} \int_A \mathbb{1}_B \, d\mathbb{P}$$

indicator function: $\mathbb{1}_B(\omega) = \begin{cases} 1, & \omega \in B \\ 0, & \omega \notin B \end{cases}$

Example: $X \sim \text{NORMAL}(0, 1^2)$, $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$,

$B = \{X > 0\}$



$$\mathbb{E}(X \mid B) = \frac{1}{\mathbb{P}(B)} \int_\Omega \underbrace{X(\omega)}_{x} \mathbb{1}_B(\omega) \, d\mathbb{P}(\omega) = \frac{1}{\mathbb{P}(B)} \cdot \int_\mathbb{R} x \underbrace{\mathbb{1}_B(X^{-1}(x))}_{= \begin{cases} 1, & x > 0 \\ 0, & x \le 0 \end{cases}} f_X(x) \, dx$$

$$= \frac{1}{\mathbb{P}(B)} \cdot \int_0^\infty x f_X(x) \, dx = 2 \cdot \frac{1}{\sqrt{2\pi}} \int_0^\infty x e^{-\frac{1}{2}x^2} \, dx = \frac{2}{\sqrt{2\pi}} \underbrace{\left(-e^{-\frac{x^2}{2}} \Big|_0^\infty \right)}_{=1}$$

General example: $\mathbb{E}(\mathbb{1}_A \mid B) = \int_\Omega \mathbb{1}_A \, d\mathbb{P}(\cdot \mid B) = \int_A d\mathbb{P}(\cdot \mid B) = \mathbb{P}(A \mid B)$

Example: Throw one die: $X : \Omega \longrightarrow \mathbb{R}$, $B = \{X = 5, X = 6\}$

$$\mathbb{E}(X \mid B) = \frac{1}{\mathbb{P}(B)} \cdot \int_B X \, d\mathbb{P} = \frac{1}{\mathbb{P}(B)} \sum_{x = 5,6} x \cdot \mathbb{P}(X = x)$$

$$= \frac{1}{\frac{2}{6}} \cdot \left(5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} \right) = \frac{11}{2} = 5.5$$