The Bright Side of Mathematics

Probability Theory – Part 19

Definition: $(\Omega, \mathcal{A}, \mathbb{P})$ probability space, $X, Y : \Omega \longrightarrow \mathbb{R}$ random variables ($\mathbb{E}(X^2), \mathbb{E}(Y^2)$ are finite)

$$\operatorname{Cov}(X,Y) := \mathbb{E}\big((X - \mathbb{E}(X))\cdot(Y - \mathbb{E}(Y))\big)$$

$$= \mathbb{E}\big(XY - X\cdot\mathbb{E}(Y) - Y\,\mathbb{E}(X) + \mathbb{E}(X)\mathbb{E}(Y)\big)$$

$$\overset{\text{linearity}}{=} \mathbb{E}(XY) - 2\cdot\mathbb{E}(Y)\mathbb{E}(X) + \mathbb{E}(X)\mathbb{E}(Y)$$

$$= \mathbb{E}(XY) - \mathbb{E}(Y)\mathbb{E}(X)$$

is called the covariance of X and Y.

Remember: X, Y independent $\Longleftrightarrow$ $\operatorname{Cov}(X,Y) = 0$ (X, Y uncorrelated)

only in special situations
(for example: X, Y normally distributed)

Property: $\operatorname{Cov}(X,Y)^2 \leq \operatorname{Cov}(X,X)\operatorname{Cov}(Y,Y)$

Definition: $\rho_{X,Y} := \dfrac{\operatorname{Cov}(X,Y)}{\sigma(X)\sigma(Y)} \in [-1,1]$ correlation coefficient

Example: $\Omega = \{a,b,c\}$, $\mathbb{P}$ uniform on Ω $\left(\mathbb{P}(\{a\}) = \mathbb{P}(\{b\}) = \mathbb{P}(\{c\}) = \frac{1}{3}\right)$

$X, Y : \Omega \longrightarrow \mathbb{R}$,

$X(a) = 1 \quad X(b) = 0 \quad X(c) = -1$

$Y(a) = 0 \quad Y(b) = 1 \quad Y(c) = 0$

$$\Rightarrow X\cdot Y = 0, \ \mathbb{E}(X) = 0 \quad \Rightarrow \operatorname{Cov}(X,Y) = 0$$

Independence? $\mathbb{P}(X \leq x, Y \leq y) = \mathbb{P}(X \leq x)\cdot\mathbb{P}(Y \leq y)$ for all x, y

$x = -1$, $y = 0$: $\mathbb{P}(\{c\}) = \mathbb{P}(\{c\})\cdot\mathbb{P}(\{a,c\})$ ↯