The Bright Side of Mathematics

Probability Theory – Part 8

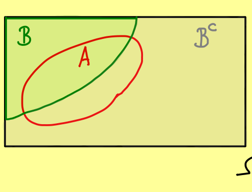
Bayes's theorem: $\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$, $\mathbb{P}(B|A) = \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)}$

$$\Rightarrow \quad \boxed{\mathbb{P}(A|B) \cdot \mathbb{P}(B) = \mathbb{P}(B|A) \cdot \mathbb{P}(A)}$$

Law of total probability: $(\Omega, \mathcal{A}, \mathbb{P})$ probability space

$A \in \mathcal{A}$

$\mathbb{P}(A)$



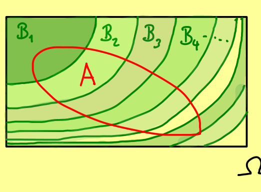
$B \in \mathcal{A}$

$\underbrace{B \cup B^c}_{\text{disjoint union}} = \Omega$

$$\mathbb{P}(A) = \mathbb{P}(\underbrace{(A \cap B) \cup (A \cap B^c)}_{\text{disjoint union}}) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c)$$

$$= \mathbb{P}(A|B) \cdot \mathbb{P}(B) + \mathbb{P}(A|B^c) \cdot \mathbb{P}(B^c)$$

Case with countably many sets: $B_i \in \mathcal{A}$ for $i \in I \subseteq \mathbb{N}$ with $\underbrace{\bigcup_{i \in I} B_i}_{\text{disjoint union}} = \Omega$



$$\mathbb{P}(A) = \mathbb{P}\Big(\underbrace{\bigcup_{i \in I} (A \cap B_i)}_{\text{disjoint union}}\Big) = \sum_{i \in I} \mathbb{P}(A \cap B_i) = \sum_{i \in I} \mathbb{P}(A|B_i) \cdot \mathbb{P}(B_i)$$

Example: Monty Hall problem

1	2	3

3 doors:
- 1 car
- 2 goats

First: You pick a door (1)

Second: Show master opens a door with a goat (never the door you picked)

Third: Stay or switch

$C_j :=$ car is behind door j , $S_j :=$ show master opens door j (in the second step)

We know: $\mathbb{P}(S_3|C_3) = 0$, $\mathbb{P}(S_3|C_2) = 1$, $\mathbb{P}(S_3|C_1) = \frac{1}{2}$

$$\mathbb{P}(C_2|S_3) \underset{\text{Bayes's theorem}}{=} \frac{\mathbb{P}(S_3|C_2) \cdot \mathbb{P}(C_2)}{\mathbb{P}(S_3)} \underset{\text{Law of total probability}}{=} \frac{\mathbb{P}(S_3|C_2) \cdot \mathbb{P}(C_2)}{\sum_{j=1}^{3} \mathbb{P}(S_3|C_j) \cdot \mathbb{P}(C_j)}$$

$$= \frac{\overbrace{\mathbb{P}(S_3|C_2)}^{=1} \cdot \overbrace{\mathbb{P}(C_2)}^{\frac{1}{3}}}{\underbrace{\mathbb{P}(S_3|C_1)}_{=\frac{1}{2}} \cdot \underbrace{\mathbb{P}(C_1)}_{=\frac{1}{3}} + \underbrace{\mathbb{P}(S_3|C_2)}_{=1} \cdot \underbrace{\mathbb{P}(C_2)}_{=\frac{1}{3}} + \underbrace{\mathbb{P}(S_3|C_3)}_{=0} \cdot \underbrace{\mathbb{P}(C_3)}_{=\frac{1}{3}}} = \frac{2}{3}$$