

Expectation and Change-of-variables

Exercise 1. Change-of-variables formula

(a) Write down and memorize the change-of-variables formula from the video for the cases $A = \mathbb{R}$ and $A = X(\Omega)$.

(b) Let $X: \Omega \rightarrow \mathbb{R}$ be a random variable with $X \sim \text{UNIFORM}([0, 2])$ and probability density f_X . Furthermore, let $g(x) = x^2$. Which of the following notations is correct according to the lecture:

(i) $\int g(X) \, d\mathbb{P}$

(ii) $\int_{\Omega} X^2 \, d\mathbb{P}$

(iii) $\int_{\mathbb{R}} \frac{x^2}{2} \mathbf{1}_{[0,2]}(x) \, dx$

(iv) $\int_0^2 \frac{x^2}{2} \, dx$

(v) $\int_{X(\Omega)} g(x) \, d\mathbb{P}_X(x)$

(vi) $\int_{\mathbb{R}} g(x) f_X(x) \, dx$

(vii) $\int_{\mathbb{R}} g(x) \, d\mathbb{P}$

(viii) $\mathbb{E}(x^2)$

(ix) $\mathbb{E}(g(X))$

(x) $\mathbb{E}(g(x))$

(xi) $\int_{\Omega} X^2 \, d\mathbb{P}_X$

(xii) $\int_0^2 X^2 \, dx$

(c) Let $X: \Omega \rightarrow \mathbb{R}$ a random variable and $A \subseteq \mathbb{R}$. Show that $\mathbb{E}(\mathbf{1}_A(X)) = \mathbb{P}_X(A)$.

(d) Let $X \sim \text{UNIFORM}([-1, 1])$. Find a function $g: \mathbb{R} \rightarrow \mathbb{R}$ such that $\int_{\Omega} g(X) \, d\mathbb{P} = \int_{-1}^1 \frac{s}{2} \, ds$.

(e) Let $X \sim \text{UNIFORM}(\{1, 2, 3, 4\})$ and $g(x) = (x - 2.5)^2$. Calculate $\mathbb{E}(g(X)) = \int_{\Omega} g(X) \, d\mathbb{P}$.

(f) Let $X \sim \text{EXP}(1)$ and $A = X^{-1}([0, 1])$. Calculate $\mathbb{E}(X \cdot \mathbf{1}_A)$.