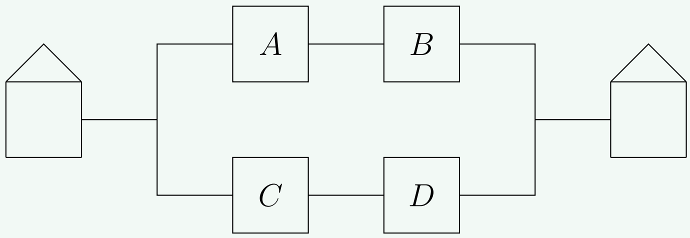
Independence for Random Variables

Exercise 1. Indepencence of Random Variables

Consider two cities connected by 4 bridges A, B, C and D. From one city one can travel to the other city if the bridges (A and B) or (C and D) are intact.



Let T be the random variable marking the point in time where one cannot travel anymore between the cities. The point in time at which a single bridge stops being intact is modelled by a random variable T_k, $k \in \{A, B, C, D\}$, where T_k is exponentially distributed with parameter α, i.e. $\mathbb{P}_{T_k}$ is the exponential distribution $\textsc{Exp}(\alpha)$.

We want to show that $\mathbb{P}(T < t) = (1 - \mathrm{e}^{-2\alpha t})^2 \quad (t \geq 0)$.

a) *If bridge A breaks after 5, B after 6, C after 7 and D after 8 years. When does route A-B stop being intact? At which time can't you travel between the two cities anymore?*

b) *Write T as a function of $T_A, \dots, T_D$.*

   Hint: The lifetime of the subsystem (A and B) can be modelled via the random variable $T_{AB} := \min(T_A, T_B)$. Something similar holds for the subsystem (C and D).

c) *Show that $\mathbb{P}(\min(T_A, T_B) \geq t) = \mathbb{P}(\{T_A \geq t\} \cap \{T_B \geq t\})$ and use this identity to calculate $\mathbb{P}(\min(T_A, T_B) < t)$. Use that the preimages $\{T_A \geq t\}$ and $\{T_B \geq s\}$ are independent for all $s, t \in \mathbb{R}$.*

d) *Use (a) and (b) to calculate $\mathbb{P}(\max(T_{AB}, T_{CD}) < t)$. Use that the preimages $\{T_{AB} < t\}$ and $\{T_{CD} < s\}$ are independent for all $s, t \in \mathbb{R}$.*