Random variables

Exercise 1. Random Variables

1. *Consider the sample space* $\Omega := \{1,2,3,4\}^2$ *of the observation of throwing two different 4-sided dice (tetrahedrons). The corresponding probability measure* $\mathbb{P}$ *is given via the uniform distribution* $\textsc{Uniform}(\Omega)$. *Furthermore, consider the random variable* $X\colon \Omega \to \mathbb{R}, \quad (\omega_1,\omega_2) \mapsto |\omega_1 - \omega_2|$ *with cumulative distribution function* F_X.

   a) *Which aspect of the experiment is observed by* X? *Specify the image* $X(\Omega)$.

   b) *Describe with you own words the event* $A = \{\omega \in \Omega \colon X(\omega) = 0\}$. *List all elements in* A.

   c) *Which of the following expressions make sense? Which notation is correct?*

      i. $\mathbb{P}(X \le 2)$
      ii. $\mathbb{P}(0)$
      iii. $\mathbb{P}(\{\omega \in \Omega \colon X(\omega) = 0\})$
      iv. $\mathbb{P}_X((-\infty, 2])$
      v. $\mathbb{P}(\{0\})$
      vi. $\mathbb{P}_X(X \le 2)$
      vii. $\mathbb{P}_X(0)$
      viii. $\mathbb{P}(X^{-1}(\{1\}))$
      ix. $F_X(X)$
      x. $\mathbb{P}(X = 0)$
      xi. $\mathbb{P}_X(\{0\})$
      xii. $F_X(2)$

   d) *Specify the probability mass function* $(p_k)_{k \in X(\Omega)}$ *corresponding to the distribution* $\mathbb{P}_X$. *Note that* $p_k := \mathbb{P}_X(\{k\})$. *Then evaluate those expressions of part 1c that have correct notation.*

2. *Consider the probability space* $(\Omega, \mathcal{A}, \mathbb{P})$ *with* $\Omega = [-1,1]$, $\mathcal{A} = \mathcal{B}([-1,1])$ *and for the probability measure* $\mathbb{P} = \textsc{Uniform}([-1,1])$.

   a) *Show that the indicator function* $\mathbf{1}_{[0,1]}\colon \Omega \to \mathbb{R}$ *is a random variable and calculate* $\mathbb{P}(\mathbf{1}_{[0,1]} = 1)$ *and* $\mathbb{P}(\mathbf{1}_{[0,1]} = 0)$. *Determine* p *such that* $\mathbb{P}_{\mathbf{1}_{[0,1]}} = \textsc{Bernoulli}(1,p)$.

   b) *Show that the random variables* $\mathbf{1}_{[0,1]}$ *and* $\mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]}$ *are independent. Are* $[0,1]$ *and* $[-\frac{1}{2},\frac{1}{2}]$ *independent in* $(\Omega, \mathcal{A}, \mathbb{P})$?

   c) *Now consider the random variable* $Y := \mathbf{1}_{[0,1]} + \mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]}$. *Determine* n *and* p *such that for the distribution of* Y *we have* $\mathbb{P}_Y = \textsc{Bin}(n,p)$.