Independence

Exercise 1. Independence

1. *Take the probability space* $([0,\infty), \mathcal{B}([0,\infty)), \textsc{Exp}(1))$ *and the events*

$$A := [\ln(1),\ \ln(2)\,], \quad B := [\ln\left(\frac{4}{3}\right),\ \ln(4)\,], \quad C := [\ln\left(\frac{16}{15}\right),\ \ln\left(\frac{16}{11}\right)\,]\,.$$

   Decide whether A, B *and* C *are (a)* independent *or (b)* pairwise independent.

2. *Take the probability space* $([0,1], \mathcal{B}([0,1]), \textsc{Uniform}([0,1]))$. *Let* A *and* B *be intervals in* $[0,1]$. *Show that if* A *and* B *are independent then* $\int_0^1 \mathbf{1}_{A\cap B}(x)\,\mathrm{d}x = \int_0^1 \mathbf{1}_A(x)\,\mathrm{d}x \cdot \int_0^1 \mathbf{1}_B(x)\,\mathrm{d}x$.

Exercise 2. Properties of Conditional Probabilities and Independence

For a general probability space $(\Omega, \mathcal{A}, \mathbb{P})$ *and* $A, B \in \mathcal{A}$, *show that the next statements are true:*

1. *If* $0 < \mathbb{P}(B) < 1$ *and* $\mathbb{P}(A \cap B) \neq 0$, *then* $\mathbb{P}(A \cap B) \neq \mathbb{P}(A \mid B)$.
2. *If* $0 < \mathbb{P}(A) < 1$, *then* A *and* A *are not independent.*
3. *If* A *and* B *are independent, then* $\complement A$ *and* B *are independent.*
4. *If* A *and* B *are independent and* $\mathbb{P}(B) > 0$, *then* $\mathbb{P}(A \mid B) = \mathbb{P}(A)$.

Show that the following statements are false in general. Also state under which additional assumptions they are true.

5. *It always holds that* $\mathbb{P}(A \mid B) = \mathbb{P}(B \mid A)$.
6. *If* $A \subseteq B$, *then* $\mathbb{P}(A \mid B) = \mathbb{P}(A)$.

Exercise 3. Independence

Let A *and* B *be two events such that*

$$\mathbb{P}(A \mid B) = \frac{1}{2}, \quad \mathbb{P}(B \mid \complement A) = \frac{1}{4}, \quad \mathbb{P}(B) = \frac{1}{4}.$$

Calculate the probability of the event A. *Furthermore, check whether* A *and* B *are independent.*