# Conditional probability measures

**Exercise 1. Conditional Probabilities**

1. *Consider the following random experiment:*

   My favourite random experiment is watching a football game and counting the number of goals where the sample space would be $\Omega = \{0, 1, 2, 3, \dots\}$.

   *In the matches of a soccer team in a season, a total of 91 goals were shot. This corresponds to a frequency of around* $\lambda = 2.68 \approx \frac{91}{34}$ *goals per match.*

   *On the corresponding probability space* $(\Omega, \mathcal{P}(\Omega), \textsc{Poi}(\lambda))$*, calculate the following (conditional) probabilities for the number of goals in a soccer match of SV Darmstadt 98:*

   a) *Two goals.* b) *At least one goal.*
   c) *The match finished with a draw (in our sample space* $\Omega$*, this corresponds to an even number of goals) given that Darmstadt shot two goals.*
   d) *Two goals, given that at least one goal was shot.*
   e) *At least one goal, given that two goals were shot.*

2. a) *Consider the following urn model: An urn contains a total of* 10 *Balls.* 5 *balls are either* ***R****ed or* ***G****reen.* 5 *balls are* ***B****lue. We take an ordered sample from this urn but draw without replacement. Let* $C = \{\boldsymbol{R}, \boldsymbol{G}, \boldsymbol{B}\}$ *denote the set of colors and, for* $\Omega = C^2 = C \times C$*, let* $(\Omega, \mathcal{P}(\Omega), \mathbb{P})$ *the probability space corresponding to the above urn model.*

   *Assume that* $\mathbb{P}(C \times \{\boldsymbol{R}\} \mid \{\boldsymbol{B}\} \times C) = \frac{10}{45}$ *and* $\mathbb{P}(C \times \{\boldsymbol{R}\} \mid \{\boldsymbol{R}, \boldsymbol{G}\} \times C) = \frac{8}{45}$*. Calculate* $\mathbb{P}(C \times \{\boldsymbol{R}\})$ *and* $\mathbb{P}(\{\boldsymbol{B}\} \times C \mid C \times \{\boldsymbol{R}\})$*.*

   b) *Consider the continuous uniform distribution on the interval* $[1, 5]$*.*

      i. *Specify a* $\sigma$*-algebra* $\mathcal{A}$ *and a probability measure* $\mathbb{P}$ *such that* $([1, 5], \mathcal{A}, \mathbb{P})$ *becomes a suitable probability space.*

      ii. *State a probability density function on* $\mathbb{R}$ *that corresponds to* $\textsc{Uniform}([1, 5])$*.*

      *Consider the event* $B = [\frac{7}{4}, \frac{9}{4})$*.*

      iii. *Calculate* $\mathbb{P}(B)$ *and* $\mathbb{P}([1, 2) \mid B)$*.*

      iv. *Calculate* $\mathbb{P}([a, b) \mid B)$ *for* $a, b \in [1, 5]$ *with* $a < b$*.*

      v. *Let* $\mathbb{P}_B$ *denote the conditional probability measure corresponding to the event* $B$*.*

      *Calculate* $\mathbb{P}_B([\frac{3}{2}, 3))$ *and* $\mathbb{P}_B([2, 3) \mid [\frac{3}{2}, 2))$*.*

   c) *Consider a discrete probability space* $(\Omega, \mathcal{P}(\Omega), \mathbb{P})$ *and a function* $f \colon \Omega \to \mathbb{N}$*. For a* ***fixed*** *set* $A \subseteq \Omega$*, assume that* $\mathbb{P}(A \mid f^{-1}(\{k\})) = \frac{1}{4}$ *and* $\mathbb{P}(f^{-1}(\{k\}) = \frac{1}{2^k}$ *for all* $k \in \mathbb{N}$*.*

      *Calculate* $\mathbb{P}(A)$ *and* $\mathbb{P}(f^{-1}(\{1\}) \mid A)$*.*