Product Probability Spaces

Exercise 1. Product Models and Product Measures

1. *We throw a die (multiple times) and are interested in the event that we throw a six. A suitable probability space looks as follows: Take* $\Omega_0 := \{1, 2, \dots, 6\}$ *for the sample space,* $\mathcal{A}_0 := \{\emptyset, \{6\}, \complement\{6\}, \Omega_0\}$ *as* σ*-algebra and create the probability space* $(\Omega_0, \mathcal{A}_0, \mathbb{P}_0)$*, where* $\mathbb{P}_0(A) := |A|/|\Omega_0|$ *for all* $A \in \mathcal{A}_0$*.*

   a) *Calculate* $\mathbb{P}_0(A)$ *for all* $A \in \mathcal{A}_0$*.*

   b) *Now consider the product* $(\Omega, \mathcal{A}, \mathbb{P})$*, where* $\Omega = \Omega_0 \times \Omega_0 \times \Omega_0$*,* $\mathcal{A}$ *is the corresponding product* σ*-algebra and* $\mathbb{P}$ *the product measure on* $\mathcal{A}$*. Describe the following events by a product of elements from* $\mathcal{A}_0$ *and calculate their probability:*

      i. *At the first throw, we have a six.*

      ii. *At the first and the second throw, we have a six.*

      iii. *At the third throw, we have a six for the first time.*

   c) *Now consider the countable product* $\Omega = \prod_{k=1}^{\infty} \Omega_0 = \Omega_0 \times \Omega_0 \times \dots$ *with corresponding product* σ*-algebra* $\mathcal{A}$ *and product measure* $\mathbb{P}$*. Formulate the events from part 1b in this setting and calculate their probability. Additionally, for* $k \in \mathbb{N}$*, let* A_k *be the event that at the* k*th throw we have a six for the first time. Also describe* A_k *as a product of elements from* $\mathcal{A}_0$ *and calculate* $\mathbb{P}(A_k)$*.*

   d) *In the setting of 1c, we define* $p_n := \mathbb{P}(A_{n+1})$ *for all* $n \in \mathbb{N}_0$*. Show that* $(p_n)_{n \in \mathbb{N}_0}$ *is a probability mass function on the sample space* $\mathbb{N}_0$*.*

      Hint: Geometric distribution.

2. *Consider the probability spaces* $(\Omega_k, \mathcal{A}_k, \mathbb{P}_k)$ *for* $k = 1, 2, 3$ *where* $\Omega_k = \mathbb{R}$*,* $\mathcal{A}_k = \mathcal{B}(\mathbb{R})$ *and* $\mathbb{P}_k = \textsc{Exp}(k)$*. Let* $(\Omega, \mathcal{A}, \mathbb{P})$ *be the corresponding product space, i.e.* $\Omega = \prod_{k=1}^{3} \Omega_k$*,* $\mathcal{A}$ *is the product* σ*-algebra and* $\mathbb{P}$ *the product measure. Calculate the probabilities of the following events*

   a) $[0, 1] \times \mathbb{R} \times \mathbb{R}$

   b) $[0, 1] \times \mathbb{R} \times \{0\}$

   c) $[0, 1] \times [1, 2] \times [2, 3]$

   d) $\mathbb{R} \times \{(a, b) \colon a + b \leq 2\}$

   Hint for 2d: Note that for $A \subseteq \Omega_a \times \Omega_b$ in the product σ-algebra we may calculate the probability of A via $\mathbb{P}(A) = \displaystyle\int_A f_a(x) \cdot f_b(y)(x, y)$.