Discrete and continuous probability measures

Exercise 1. Probability Measures

1. *Let $(\Omega, \mathcal{A})$ be an arbitrary event space. For **fixed** $\omega \in \Omega$, we define the mapping*

$$\mathbb{P}_\omega(A) := \begin{cases} 1 & \omega \in A, \\ 0 & \omega \notin A. \end{cases} \qquad (A \in \mathcal{A}) \qquad (*)$$

   a) *For the event space $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ and the fixed value $\omega = 5$ calculate*

$$\mathbb{P}_5(\emptyset), \qquad \mathbb{P}_5(\{1,2,3\}),$$
$$\mathbb{P}_5(\{n \in \mathbb{N} \colon n \text{ is a prime number}\}), \qquad \mathbb{P}_5(\{5\})\ .$$

   b) *Show that $\mathbb{P}_5$ defines a probability measure on $\mathcal{P}(\mathbb{N})$.*

2. *Consider the probability space $([0,1], \mathcal{B}([0,1]), \lambda)$ where the geometric volume measure λ is defined on half-open intervals via*

$$\lambda((a,b]) := b - a, \quad 0 \le a < b \le 1.$$

   a) *Show that $\{1\} = \bigcap_{n \in \mathbb{N}} (1 - \frac{1}{n}, 1]$. Use a property of σ-algebras to show that $\{1\} \in \mathcal{B}([0,1])$.*

   b) *Use 2a and properties of probability measures to calculate $\lambda(\{1\})$. Also state the value of $\lambda(\{x\})$ for arbitrary $x \in [0,1]$.*

   c) *Use 2b and further properties of probability measures to calculate*

$$\lambda\Big((0,\tfrac{1}{3}] \cup (\tfrac{2}{3},1]\Big), \qquad \lambda\Big([0,\frac{1}{2}]\Big), \qquad \lambda\Big([0,\frac{1}{2})\Big),$$
$$\lambda\Big((0,\frac{1}{2}] \setminus [\frac{1}{4},\frac{3}{4}]\Big), \qquad \lambda\Big(\bigcup_{n \in \mathbb{N}} \{\frac{1}{n}\}\Big)\ .$$

   Start by showing that each of the events involved is an element of the σ-algebra $\mathcal{B}([0,1])$.

3. a) *Consider the sample space $\Omega = \{$'C', 'J', 'F'$\}$ and define the event space via $(\Omega, \mathcal{P}(\Omega))$. Let $\mathbb{P}$ be the* probability measure *defined via $\mathbb{P}(A) = |A|/|\Omega|$ for all $A \in \mathcal{P}(\Omega)$. State the corresponding* probability mass function.

   b) *Consider the event space $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$. Determine $c \in \mathbb{R}$ such that $p_k = c^k, k \in \mathbb{N}$ becomes a probability mass function on this event space. Let $\mathbb{P}$ be the corresponding probability measure. Calculate*

$$\mathbb{P}(\{1\}), \quad \mathbb{P}(\{1,2,3\}), \quad \mathbb{P}(\{1,3,5\}),$$
$$\mathbb{P}\Big(\bigcup_{n \in \mathbb{N}} \{2, \ldots, 2+n\}\Big), \quad \mathbb{P}(\{n \in \mathbb{N} \colon n \text{ is even}\})\ .$$

c) *Consider the event space* $([0,2], \mathcal{B}([0,2]))$ *and the event* $A = [0,1] \cup [\frac{3}{2}, 2]$. *For those of the following functions that are* probability density functions, *calculate the probability of the event* A.

$$f(x) = \frac{1}{4}\mathbf{1}_{[0,1]}(x) + \frac{3}{4}\mathbf{1}_{[1,2]}(x), \qquad g(x) = x - \frac{1}{2}, \qquad h(x) = \cos\left(\frac{\pi}{4}x\right).$$