Starting with events and sigma-algebras

Exercise 1. Sample Space

For each the following examples of random experiments, *describe one corresponding* sample space Ω *(there may be more than one sample space that makes sense). Give an example of an element $\omega \in \Omega$ and calculate the total number of elements $|\Omega|$.*

(a) A simple coin toss.

(b) We throw a coin infinitely many times.

(c) We throw a coin infinitely many times and count the number of observed heads.

(d) Throwing a 6-sided die twice.

(e) Throwing a 6-sided die twice and calculating the eye-sum.

(f) We observe the sky at night and measure the time (in minutes) until we see a shooting star.

(g) A company produces metal tubes of length $l > 0$. At quality control, the deviation of the length of the probe from l is determined. A negative deviation means that the probe is smaller than l.

Exercise 2. Events

Give a set-theoretical description of the following events in your sample spaces from Problem 1:

1. *In your sample spaces from d and e, consider the events that*
   a) *the eye-sum is less or equal to 12,*
   b) *the eye-sum is divisible by 5,*
   c) *the eye-sum is equal to 1.*
2. *In c, consider the event that the number of observed heads is non-zero, smaller or equal than 42 and (divisible by (3 and 5) or divisible 7).*
3. *In f, consider the event that we see a shooting star in less than 10 minutes.*
4. *In g, consider the event that the deviation is smaller than 1 and greater than $-\frac{1}{5}$.*
5. *In the Cartesian product of your sample spaces from a and d, consider the event that the coin shows tails and the die shows 5 in the first and a number between 3 and 6 in the second throw.*

Exercise 3. σ-Algebras

1. *Let Ω be a sample space and $A \subset \Omega$ with $A \neq \Omega$ an event. Which of the following families of sets is a σ-algebra? Which of the σ-algebras contains the*

event A?

$$\mathcal{A}_1 := \{\emptyset, \Omega\} \quad \mathcal{A}_2 := \{\emptyset, A, \Omega\} \quad \mathcal{A}_3 := \{\emptyset, A, \complement A\} \quad \mathcal{A}_4 := \mathcal{P}(\Omega)$$

2. *Consider the sample space* $\Omega := \{1, 2, 3\}$. *List all* σ*-algebras on* Ω.

   Hint: Every σ-algebra on a finite sample space Ω corresponds to a partition of Ω.

3. *Let* $\mathcal{A}$ *be a* σ*-algebra on the sample space* Ω *as defined in the lecture notes. Which of the following statements are true and why (not)?*

   a) $\Omega \in \mathcal{A}$

   b) $A \in \mathcal{A} \implies \complement A \in \mathcal{A}$

   c) $(A_n)_{n\in\mathbb{N}}$ *in* $\mathcal{A} \implies \bigcap_{n\in\mathbb{N}} A_n \in \mathcal{A}$

   d) $A, B \subseteq \Omega \implies A \cap B \in \mathcal{A}$

4. *Consider two sample spaces* Ω_1 *and* Ω_2 *and, for* $i = 1, 2$, *two events* $A_i \subseteq \Omega_i$. *Let us denote by* $\mathcal{A}_i$ *the smallest* σ*-algebra on* Ω_i *that contains* A_i, *cf. lecture notes. Now, list all elements of the product* σ*-algebra* $\mathcal{A}$ *on the sample space* $\Omega := \Omega_1 \times \Omega_2$.

Exercise 4. Recap – Indicator Functions and Integration

For fixed $A \subseteq \mathbb{R}^d$, *we define the indicator function* $\mathbf{1}_A : \mathbb{R}^d \to \{0, 1\}$ *via*

$$\mathbf{1}_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

1. *Sketch the functions* $\mathbf{1}_{[0,1]}$, $\mathbf{1}_{[0,\infty)}$ *and* $\mathbf{1}_{\mathbb{N}}$.

2. *Show that, for fixed* $y \in \mathbb{R}$, *we have that* $\mathbf{1}_{[a,b]}(x - y) = \mathbf{1}_{[a+y,b+y]}(x)$.

3. *Show that, for* $A, B \subset \mathbb{R}^d$ *and* $x, y \in \mathbb{R}^d$, *we have that*

   a) $\mathbf{1}_{A\cap B}(x) = \mathbf{1}_A(x) \cdot \mathbf{1}_B(x)$,

   b) $\mathbf{1}_{A\cup B}(x) = \max\{\mathbf{1}_A(x), \mathbf{1}_B(x)\}$,

   c) $\mathbf{1}_{\complement A}(x) = 1 - \mathbf{1}_A(x)$ *and*

   d) $\mathbf{1}_{A\times B}(\,(x, y)\,) = \mathbf{1}_A(x) \cdot \mathbf{1}_B(y)$.

4. *Calculate the following integrals*

   a) $\displaystyle\int_{-3}^{3} \mathbf{1}_{[-1,1]}(x)\,\mathrm{d}x$ b) $\displaystyle\int_{-3}^{3} x\,\mathbf{1}_{[-1,1]}(x)\,\mathrm{d}x$ c) $\displaystyle\int_{0}^{3} \mathbf{1}_{[-1,1]}(x)\,\mathrm{d}x$.

5. *For* $x \in \{-2, 0, \frac{1}{2}\}$, *calculate the integral* $\displaystyle\int_{-3}^{3} \mathbf{1}_{[-1,1]}(x - y) \cdot \mathbf{1}_{[-1,1]}(y)\,\mathrm{d}y$.

6. *Calculate the integral* $\displaystyle\int_0^1 \left(\int_1^2 \mathbf{1}_{[0,\frac{1}{2}]\times[1,\frac{3}{2}]}(\,(x, y)\,)\,\mathrm{d}x \right) \mathrm{d}y$.

7. *Let* $C := \{(x,y) \in \mathbb{R}^d \colon x < y\}$. *Calculate the integral* $\displaystyle\int_{[0,1]\times[0,1]} \mathbf{1}_C(x,y)\,\mathrm{d}(x,y)$.