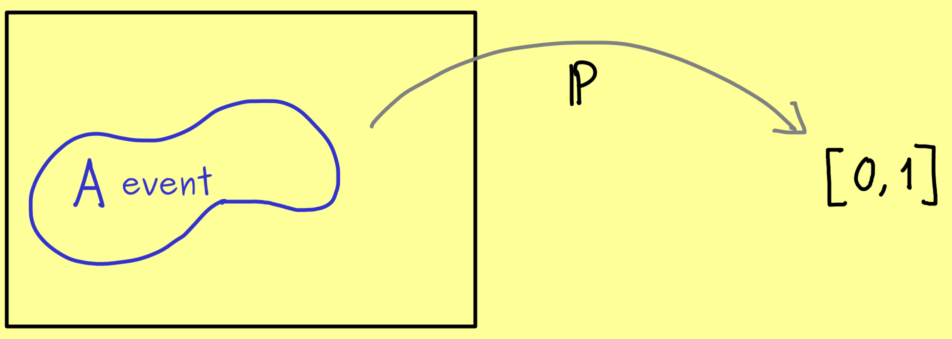
The Bright Side of Mathematics

Probability Theory – Part 3

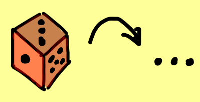
events form σ-algebra



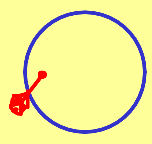
Ω sample space

discrete case | (absolutely) continuous case | mixed and other cases

"finitely many outcomes"
"countably many outcomes"



"uncountably many outcomes"



σ-additivity: $\mathbb{P}\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mathbb{P}(A_j)$

if we have pairwise disjoint sets

discrete

sample space Ω finite or countable set

(Example: $\Omega = \{\text{Heads}, \text{Tails}\}$, $\Omega = \mathbb{N}$)

σ-algebra $\mathcal{A} = \mathcal{P}(\Omega)$

probability measure $\mathbb{P}: \mathcal{A} \longrightarrow [0,1]$

is completely determined by $\mathbb{P}(\{\omega\})$ for all $\omega \in \Omega$

probability mass function: $(p_\omega)_{\omega\in\Omega}$ with $p_\omega \geq 0$, $\sum_{\omega\in\Omega} p_\omega = 1$

Define: $\mathbb{P}(A) := \sum_{\omega \in A} p_\omega$

Example: $\Omega = \{1,2,3,4,5,6\}$ unfair die

$p_1 = \frac{1}{10}$ $p_2 = \frac{1}{10}$ $p_3 = \frac{1}{10}$ $p_4 = \frac{1}{10}$ $p_5 = \frac{1}{10}$ $p_6 = \frac{1}{2}$

$$\mathbb{P}(\{1,2,3,4,5\}) = \sum_{\omega=1}^{5} p_\omega = 5 \cdot \frac{1}{10} = \frac{1}{2}$$

(abs.) continuous

sample space $\Omega \subseteq \mathbb{R}^n$ uncountable, $\Omega \in \mathcal{B}(\mathbb{R}^n)$ (Borel set)

(Example: $\Omega = [0,1]$)

σ-algebra $\mathcal{A} = \mathcal{B}(\Omega)$

probability measure $\mathbb{P}: \mathcal{A} \longrightarrow [0,1]$

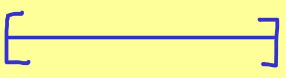
can be described by

probability density function: $f: \Omega \longrightarrow \mathbb{R}$ with $f(x) \geq 0$, $\int_\Omega f(x)\,dx = 1$

measurable!

Define: $\mathbb{P}(A) := \int_A f(x)\,dx$

Example: $\Omega = [0,2]$



throw point into interval

$f: \Omega \longrightarrow \mathbb{R}$ with $f(x) = \frac{1}{2}$

Hence: $\int_0^2 f(x)\,dx = \frac{1}{2} \cdot 2 = 1$

$$\mathbb{P}(A) = \int_A f(x)\,dx = \frac{1}{2}\int_A 1\,dx = \frac{1}{2}\,\text{Lebesgue measure}\,(A)$$

$$\mathbb{P}([a,b]) = \frac{1}{2}(b-a)$$