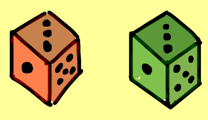
The Bright Side of Mathematics

Probability Theory – Part 10

Random variables $X : \Omega \longrightarrow \mathbb{R}$ with some properties.

Example: Throwing two dice  $\left(\underset{\{1,2,3,4,5,6\}^2}{\Omega \atop \shortparallel}, \underset{\mathcal{P}(\Omega)}{\mathcal{A} \atop \shortparallel}, \mathbb{P}\right)$ = uniform distribution

$$X : \Omega \longrightarrow \mathbb{R}, \quad (\omega_1, \omega_2) \longmapsto \omega_1 + \omega_2$$

random variable gives sum of the numbers the dice show

Definition: Let $(\Omega, \mathcal{A})$ and $(\tilde{\Omega}, \tilde{\mathcal{A}})$ be measurable spaces (= event spaces).

A map $X : \Omega \longrightarrow \tilde{\Omega}$ is called a random variable if

$$X^{-1}(\tilde{A}) \in \mathcal{A} \quad \text{for all } \tilde{A} \in \tilde{\mathcal{A}}.$$

Examples: (a) $\left(\underset{\{1,2,3,4,5,6\}^2}{\Omega \atop \shortparallel}, \underset{\mathcal{P}(\Omega)}{\mathcal{A} \atop \shortparallel}\right)$ and $\left(\underset{\mathbb{R}}{\tilde{\Omega} \atop \shortparallel}, \underset{\mathcal{B}(\mathbb{R})}{\tilde{\mathcal{A}} \atop \shortparallel}\right)$, $X : \Omega \longrightarrow \mathbb{R}, \ (\omega_1, \omega_2) \longmapsto \omega_1 + \omega_2$

$$X^{-1}(\tilde{A}) \in \mathcal{P}(\Omega) \ \text{ for all } \tilde{A} \in \tilde{\mathcal{A}}. \Rightarrow X \text{ is a random variable}$$

(b) $\left(\underset{\{1,2,3,4,5,6\}^2}{\Omega \atop \shortparallel}, \underset{\{\emptyset, \Omega\}}{\mathcal{A} \atop \shortparallel}\right)$ and $\left(\underset{\mathbb{R}}{\tilde{\Omega} \atop \shortparallel}, \underset{\mathcal{B}(\mathbb{R})}{\tilde{\mathcal{A}} \atop \shortparallel}\right)$, $X : \Omega \longrightarrow \mathbb{R}, \ (\omega_1, \omega_2) \longmapsto \omega_1 + \omega_2$

$$X^{-1}(\{2\}) = \{(1,1)\} \notin \mathcal{A} \Rightarrow X \text{ is not a random variable}$$

Notation: Let $(\Omega, \mathcal{A})$ and $(\tilde{\Omega}, \tilde{\mathcal{A}})$ be measurable spaces (= event spaces).

probability measure $\mathbb{P} : \mathcal{A} \longrightarrow [0,1]$, $X : \Omega \longrightarrow \tilde{\Omega}$ random variable

$$\mathbb{P}(X \in \tilde{A}) := \mathbb{P}\big(X^{-1}(\tilde{A})\big) = \mathbb{P}\big(\{\omega \in \Omega \mid X(\omega) \in \tilde{A}\}\big)$$