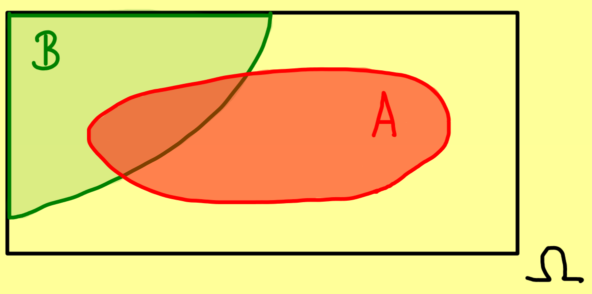
BECOME A MEMBER

ON STEADY

The Bright Side of
Mathematics

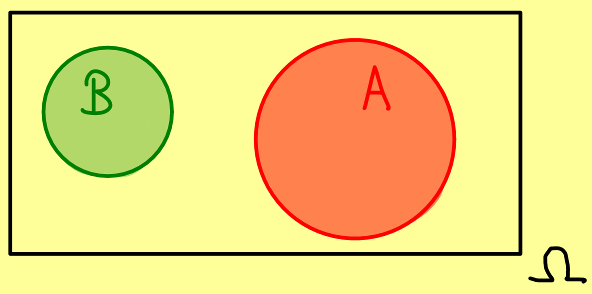
Probability Theory – Part 9

Independence (for events)



$A, B \subseteq \Omega$ events

independent?

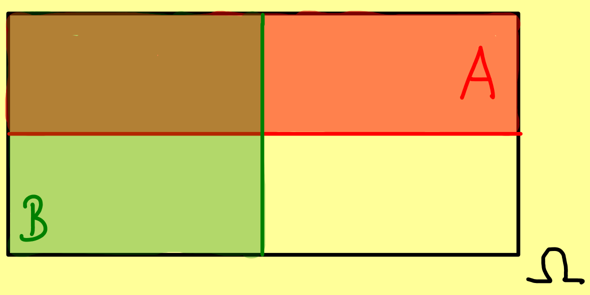


$A, B \subseteq \Omega$ events

independent!

We want: $\mathbb{P}(A \mid B) = \mathbb{P}(A)$ and $\mathbb{P}(B \mid A) = \mathbb{P}(B)$

Example:



$\mathbb{P}(A) = \frac{1}{2}$, $\mathbb{P}(A \mid B) = \frac{1}{2}$

$\mathbb{P}(B) = \frac{1}{2}$, $\mathbb{P}(B \mid A) = \frac{1}{2}$

$\Rightarrow$ independent!

Recall: $\mathbb{P}(A) \overset{!}{=} \mathbb{P}(A \mid B) = \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$, $\mathbb{P}(B) \overset{!}{=} \mathbb{P}(B \mid A) = \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}$

$\Leftrightarrow \quad \mathbb{P}(A \cap B) \overset{!}{=} \mathbb{P}(A) \cdot \mathbb{P}(B)$

Definition: Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space.

Two events $A, B \in \mathcal{A}$ are called <u>independent</u> if $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$.

A family $(A_i)_{i \in I}$ with $A_i \in \mathcal{A}$ is called <u>independent</u> if

$$\mathbb{P}\Big(\bigcap_{j \in J} A_j\Big) = \prod_{j \in J} \mathbb{P}(A_j) \quad \text{for all finite } \emptyset \neq J \subseteq I .$$

Example:

2 throws <u>with order</u>: $(\Omega, \mathcal{A}, \mathbb{P})$ with $\Omega = \{1,2,3,4,5,6\}^2$, $\mathcal{A} = \mathcal{P}(\Omega)$, $\mathbb{P}$ = uniform distribution

$\mathbb{P}(\{(\omega_1, \omega_2)\}) = \frac{1}{36}$

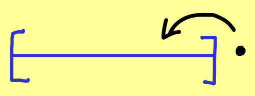
A = "first throw gives 6" $= \{(\omega_1, \omega_2) \in \Omega \mid \omega_1 = 6\}$

B = "sum of both throws is 7" $= \{(\omega_1, \omega_2) \in \Omega \mid \omega_1 + \omega_2 = 7\}$

$\mathbb{P}(A) = \frac{1}{6}$, $\mathbb{P}(B) = \mathbb{P}(\{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}) = \frac{6}{36} = \frac{1}{6}$

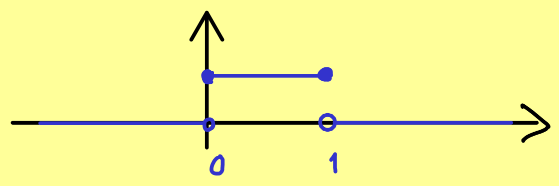
$\mathbb{P}(A \cap B) = \mathbb{P}(\{(6,1)\}) = \frac{1}{36} = \mathbb{P}(A) \cdot \mathbb{P}(B) \Rightarrow A, B$ are independent

Example:



throw a point into unit interval $(\Omega, \mathcal{A}, \mathbb{P})$ with $\Omega = [0,1]$, $\mathcal{A} = \mathcal{B}(\Omega)$, $\mathbb{P}$ = uniform distribution density function

$\mathbb{P}([a,b]) = \int_{[a,b]} 1 \, dx = b - a$ $\boxed{\text{for } b > a \text{ and } a, b \in \Omega}$ $f: \Omega \to \mathbb{R}$ with $f(x) = 1$



indicator function: $\mathbb{1}_{[0,1]}(x) := \begin{cases} 1 , & x \in [0,1] \\ 0 , & \text{else} \end{cases}$

For two independent events $A, B \in \mathcal{A}$, we have:

$$\int_{A \cap B} \mathbb{1}_{[0,1]}(x) \, dx = \mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B) = \int_A \mathbb{1}_{[0,1]}(x) \, dx \cdot \int_B \mathbb{1}_{[0,1]}(x) \, dx$$

$$\int_{[0,1]} \mathbb{1}_{A \cap B}(x) \, dx = \int_{[0,1]} \mathbb{1}_A(x) \, dx \cdot \int_{[0,1]} \mathbb{1}_B(x) \, dx$$