Dealing with Sets and Maps

Exercise 1. Sets, Products, Functions

1. *Complete the following sentences:*
   a) *The set* $A := \{x \in \mathbb{R} : (x+1) \in (-\infty, 0]\}$ *contains all* x *of* $\mathbb{R}$ *such that* ...
   b) *The set* $B := \{1 - \exp(-x) : |x| > 4\}$ *contains all* ...
   c) *The set* $C := \{(x, y) \in [0, \infty) \times \mathbb{R} : x \leq y\}$ *contains all pairs of numbers* (x, y) *where* $x \in \dots$ *and* $y \in \dots$ *such that* $\dots$.
   d) x *is an element of the set* $\bigcap_{n=1}^{\infty}[0, 1/n]$ *if* $\dots \leq \dots \leq \dots$ *for all* .... *In particular* $\dots = 0$.
   e) y *is an element of the set* $\bigcup_{n\in\mathbb{N}}[n, n+1)$ *if there exists* $\dots \in \dots$ *such that* $\dots \leq \dots < \dots$.
   f) *Let* (a_n) *and* (b_n) *be sequences in* $\mathbb{R}$ *and* $\Omega \subseteq \mathbb{R}$. *We want to prove the following equality of sets:*

   $$\bigcap_{n\in\mathbb{N}} \Big((a_n, b_n) \cap \Omega \Big) = \Big(\bigcap_{n\in\mathbb{N}} (a_n, b_n) \Big) \cap \Omega.$$

   We first show that $x \in \bigcap_{n\in\mathbb{N}} \Big((a_n, b_n) \cap \Omega \Big) \implies x \in \Big(\bigcap_{n\in\mathbb{N}} (a_n, b_n) \Big) \cap \Omega$:

      i. *Let* $x \in \bigcap_{n\in\mathbb{N}} \Big((a_n, b_n) \cap \Omega \Big)$.
      ii. *Then we have that* $x \in \dots$ *for all* $n \in \mathbb{N}$.
      iii. *This shows that* $x \in \dots$ *and* $x \in \Omega$ *for all* $n \in \mathbb{N}$.
      iv. *This implies that* $x \in \dots$ *and* $x \in \Omega$ *which allows us to conclude that the inclusion* $x \in \Big(\bigcap_{n\in\mathbb{N}} (a_n, b_n) \Big) \cap \Omega$ *holds.*

   Now, use a similar proof to show the converse statement.

2. *Define a suitable set that matches the following descriptions:*
   a) A *is the set of all open real intervals with rational endpoints.*
   b) *If* $f : \mathbb{R} \to \mathbb{R}$ *is a function, then* B *is the set of all* $x \in \mathbb{R}$ *such that* $|f(x)|$ *is smaller than* 5.
   c) C *is the set of all real numbers smaller or equal* 4 *and strictly greater than* 3.
   d) D *is the set of all vectors of length* 3 *with positive integer entries, such that the sum of the entries equals* 9.

   For each of the sets A, B, C *and* D *give an example of an element of the respective set.*

Exercise 2. Functions, Images and Preimages

(a) Let $f\colon \mathbb{R} \to \mathbb{R}$ *and* $a \in \mathbb{R}$*. Which of the following sets coincide?*

a) $f^{-1}([0,\infty))$

b) $\{f(x)\colon x \in [0,\infty)\}$

c) $\{f(x)\colon (x+a) \in [a,\infty)\}$

d) $\{x \in \mathbb{R}\colon f(x) \in [0,\infty)\}$

e) $\{f(x)\colon x \geq 0 \text{ and } x < \infty\}$

f) $\{f(x)\colon 0 \leq x < \infty\}$

g) $f(\{x \in \mathbb{R}\colon x \geq 0\})$

h) $f([0,\infty))$

i) $\{y \in f(\mathbb{R})\colon y \geq 0\}$

(b) Let $f\colon \mathbb{R} \to \mathbb{R}$ *and* $a > 0$*. Which of the following sets coincide?*

a) $f^{-1}([-a,a])$

b) $\{x \in \mathbb{R}\colon |f(x)| \leq a\}$

c) $\{x \in \mathbb{R}\colon \max\{f(x), -f(x)\} \leq a\}$

d) $f(\mathbb{R})$

e) $f^{-1}([-a,\infty)) \cap f^{-1}((-\infty,a])$

f) $\{f(x)\colon |x| \leq a\}$

g) $\{f(x)\colon |x| \in [0,a]\}$

h) $f([a,\infty)) \cap f((-\infty,a])$

i) $\{f(x)\colon x \geq -a \text{ and } x \leq a\}$

j) $\{y \in \mathbb{R}\colon \exists x \in \mathbb{R} \text{ with } f(x) = y\}$

(c) Let $f\colon X \to Y$ *and* $A \subseteq X$ *and* $B \subseteq Y$*. Which of the following sets coincide?*

a) $\{(x,y) \in X \times Y\colon \text{there exists } x \in A \text{ such that } f(x) = y\}$

b) $\{(x,y)\colon x \in X, y \in Y\}$

c) $\{(x,y) \in X \times Y : y \in f(A)\}$

d) $\{(x,y) \in X \times Y : f(x) \in B \text{ and } f(x) = y\}$

e) $f^{-1}(B) \times Y$

f) $\{(x,y) \in X \times Y\colon f(x) \in Y\}$

g) $X \times Y$

h) $\{(x,y) \in A \times Y : y \in B\}$

i) $A \times B$

j) $X \times f(A)$