The Bright Side of Mathematics

Ordinary Differential Equations – Part 10

Initial value problem: $\dot{x} = v(x)$ with $v: \mathbb{R}^n \to \mathbb{R}^n$ loc. Lipschitz continuous

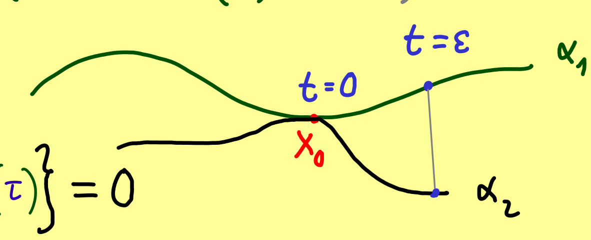
$x(0) = x_0$

Theorem: The initial value problem has at most one solution. (orbits don't cross!)

Proof: Assume α_1, α_2 are two distinct solutions $\left(\alpha_1(0) = \alpha_2(0) = x_0\right)$

with $\alpha_1(\varepsilon) \neq \alpha_2(\varepsilon)$ for $\varepsilon > 0$ and

$\inf \{\tau \in [0,\varepsilon] \mid \alpha_1(\tau) \neq \alpha_2(\tau)\} = 0$



$$\|\beta(t)\| = \|\alpha_1(t) - \alpha_2(t)\|$$

$$= \left\| \int_0^t \dot{\alpha}_1(\tau)\, d\tau - \int_0^t \dot{\alpha}_2(\tau)\, d\tau \right\| = \left\| \int_0^t v(\alpha_1(\tau))\, d\tau - \int_0^t v(\alpha_2(\tau))\, d\tau \right\|$$

$$= \left\| \int_0^t \big(v(\alpha_1(\tau)) - v(\alpha_2(\tau))\big)\, d\tau \right\| \leq \int_0^t \underbrace{\| v(\alpha_1(\tau)) - v(\alpha_2(\tau)) \|}_{\leq L \cdot \|\alpha_1(\tau) - \alpha_2(\tau)\|}\, d\tau$$

$$\leq L \cdot \int_0^\varepsilon \underbrace{\|\beta(\tau)\|}_{\leq \max \|\beta(\tau)\|}\, d\tau \leq \underbrace{L \cdot \varepsilon}_{} \cdot \max_{\tau \in [0,\varepsilon]} \|\beta(\tau)\|$$

choose ε such that $L\varepsilon \leq \frac{1}{2}$

$$\Rightarrow \boxed{\|\beta(t)\| \leq \frac{1}{2} \cdot \max_{\tau \in [0,\varepsilon]} \|\beta(\tau)\| \quad \text{for all } t \in (0,\varepsilon]} \Rightarrow \alpha_1 = \alpha_2 \;\lightning$$