Ordinary Differential Equations - Part 21

System of linear differential equations: $\dot{x} = A(t)\,x + b(t) \quad (*)$

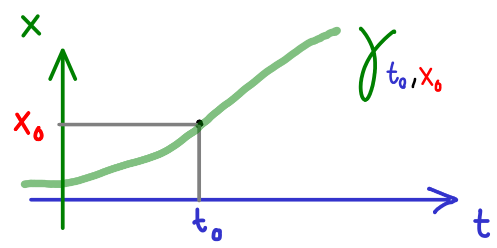
with $I \ni t \xrightarrow[\text{continuous}]{} A(t) \in \mathbb{R}^{n\times n}$

$I \ni t \longmapsto b(t) \in \mathbb{R}^n$

interval in $\mathbb{R}$

We already know:

- the homogeneous part of $(*)$ $\left(\dot{x} = A(t)\,x\right)$ has an n-dimensional solution space $\mathcal{S}_0$
- the initial value problem $(\text{IVP}^{t_0}_{x_0})$ $\boxed{\begin{array}{l}\dot{x} = A(t)\,x + b(t)\\ x(t_0) = x_0\end{array}}$ has a global solution $\gamma_{t_0,x_0} : I \longrightarrow \mathbb{R}^n$



Solution set: $\mathcal{S} := \left\{ \beta : I \to \mathbb{R}^n \text{ continuously differentiable} \;\middle|\; \beta \text{ solution of } (*) \right\}$

$$\mathcal{S}_0 + \gamma_{t_0,x_0} := \left\{ \alpha + \gamma_{t_0,x_0} \;\middle|\; \alpha \in \mathcal{S}_0 \right\} \qquad \text{(affine subspace)}$$

Show $\mathcal{S} = \mathcal{S}_0 + \gamma_{t_0,x_0}$:

$(\supseteq)$ Take $\alpha \in \mathcal{S}_0$: $A(t)\big(\alpha(t) + \gamma_{t_0,x_0}(t)\big) + b(t)$

$$= \underbrace{A(t)\,\alpha(t)} + \underbrace{A(t)\,\gamma_{t_0,x_0}(t) + b(t)}$$

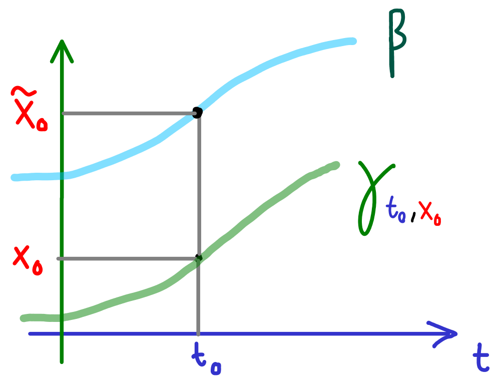
$$= \dot{\alpha}(t) + \dot{\gamma}_{t_0,x_0}(t)$$

$$= \left(\alpha + \gamma_{t_0,x_0}\right)^{\boldsymbol{\cdot}}(t)$$

$$\Longrightarrow \alpha + \gamma_{t_0,x_0} \in \mathcal{S}$$

($\subseteq$) Take $\beta \in S$ and set $\tilde{x}_0 := \beta(t_0)$

$\Rightarrow$ β is solution of $(\mathrm{IVP}^{t_0}_{\tilde{x}_0})$



Choose $\alpha \in S_0$ as the solution of the initial value problem
$$\boxed{\begin{aligned} \dot{x} &= A(t)\,x \\ x(t_0) &= \tilde{x}_0 - x_0 \end{aligned}}$$

Then: $\alpha + \gamma_{t_0,x_0} \in S$ with $(\alpha + \gamma_{t_0,x_0})(t_0) = \alpha(t_0) + \gamma_{t_0,x_0}(t_0)$

$$= \tilde{x}_0 - x_0 + x_0 = \tilde{x}_0$$

$\Rightarrow$ $\alpha + \gamma_{t_0,x_0}$ is solution of $(\mathrm{IVP}^{t_0}_{\tilde{x}_0})$

$\overset{\text{uniqueness}}{\Longrightarrow}$ $\beta = \alpha + \gamma_{t_0,x_0}$ $\square$

Result: The solution set of $\dot{x} = A(t)\,x + b(t)$ is given by

$$S = S_0 + \gamma$$

where S_0 is the solution space of the homogeneous part $\dot{x} = A(t)\,x$ and γ is a particular solution of $\dot{x} = A(t)\,x + b(t)$.

(S is an n-dimensional affine subspace)