BECOME A MEMBER

ON STEADY

The Bright Side of Mathematics

Ordinary Differential Equations - Part 24

System of linear ODEs (homogeneous + autonomous)

$\dot{x} = A x$ (A: $(n \times n)$-matrix) $\rightsquigarrow$ solutions $t \longmapsto e^{tA} \cdot x_0$ for $x_0 \in \mathbb{R}^n$

(e^{tA}:) easy to calculate if A has n different eigenvalues: $\lambda_1, \lambda_2, \ldots, \lambda_n$

Linear ODE of order n (homogeneous + autonomous)

$$x^{(n)} + a_{n-1} x^{(n-1)} + a_{n-2} x^{(n-2)} + \cdots + a_1 \dot{x} + a_0 x = 0$$

Transform into system of first order: $\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x \\ \dot{x} \\ \vdots \\ x^{(n-1)} \end{pmatrix}$ and $\dot{y}_n = -a_{n-1} y_n - \cdots - a_1 y_2 - a_0 y_1$

$$\Longrightarrow \quad \dot{y} = \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \vdots \\ \dot{y}_n \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ \vdots \\ y_n \\ \dot{y}_n \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & & -a_{n-1} \end{pmatrix}}_{A \quad (n \times n)\text{-matrix}} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

General solution: $e^{tA} \iff$ eigenvalues of A

Characteristic polynomial: $\det(A - \lambda \cdot \mathbb{1}) = (-1)^n \underbrace{\left(\lambda^n + a_{n-1} \lambda^{n-1} + a_{n-2} \lambda^{n-2} + \cdots + a_1 \cdot \lambda^1 + a_0\right)}$

is called the characteristic polynomial of the ODE

$$\boxed{x^{(n)} + a_{n-1} x^{(n-1)} + a_{n-2} x^{(n-2)} + \cdots + a_1 \dot{x} + a_0 x = 0}$$

Rule of thumb: Use approach $x(t) = e^{\lambda t}$

Result: If the characteristic polynomial has n different zeros in the real numbers

$\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$, then:

$$e^{\lambda_1 t} \cdot \begin{pmatrix} 1 \\ \lambda_1 \\ \lambda_1^2 \\ \vdots \\ \lambda_1^{n-1} \end{pmatrix}, \; e^{\lambda_2 t} \cdot \begin{pmatrix} 1 \\ \lambda_2 \\ \lambda_2^2 \\ \vdots \\ \lambda_2^{n-1} \end{pmatrix}, \; \ldots, \; e^{\lambda_n t} \cdot \begin{pmatrix} 1 \\ \lambda_n \\ \lambda_n^2 \\ \vdots \\ \lambda_n^{n-1} \end{pmatrix}$$

span solution space of $\dot{y} = A y$

and $t \mapsto e^{\lambda_1 t}$, $t \mapsto e^{\lambda_2 t}$, ..., $t \mapsto e^{\lambda_n t}$ span solution space of

$$\boxed{x^{(n)} + a_{n-1} x^{(n-1)} + a_{n-2} x^{(n-2)} + \cdots + a_1 \dot{x} + a_0 x = 0}$$