BECOME A MEMBER

ON STEADY

The Bright Side of Mathematics

Ordinary Differential Equations - Part 23

Example: System of linear differential equations (homogeneous + autonomous)

$$\dot{x}_1 = -x_1 + 3x_2 \qquad \dot{x}_2 = x_1 + x_2 \quad \Longleftrightarrow \quad \dot{x} = \underbrace{\begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix}}_{A} x$$

General solution: $e^{tA} = \sum_{k=0}^{\infty} \frac{(t \cdot A)^k}{k!}$

(columns span solution space)

Remark: If $B = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, then $B^k = \begin{pmatrix} \lambda^k & 0 \\ 0 & \mu^k \end{pmatrix}$ and $e^B = \begin{pmatrix} e^\lambda & 0 \\ 0 & e^\mu \end{pmatrix}$

If A is diagonalizable, then $A = XDX^{-1}$, $A^2 = XD\underbrace{X^{-1}X}_{\mathbb{1}}DX^{-1} = XD^2X^{-1}$

$A^k = XD^kX^{-1}$

$$\Longrightarrow \quad e^{tA} = X \cdot e^{tD} X^{-1}$$

Back to the example: $A = \begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix}$

eigenvalues: $0 = \det(A - \lambda \cdot \mathbb{1}) = \det \begin{pmatrix} -1-\lambda & 3 \\ 1 & 1-\lambda \end{pmatrix} = \lambda^2 - 4$

$\Longrightarrow \lambda_1 = -2 \,,\; \lambda_2 = 2$

eigenvectors: $\operatorname{Ker}(A - \lambda_1 \cdot \mathbb{1}) = \operatorname{Ker} \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} = \operatorname{Span} \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

$\operatorname{Ker}(A - \lambda_2 \cdot \mathbb{1}) = \operatorname{Ker} \begin{pmatrix} -3 & 3 \\ 1 & -1 \end{pmatrix} = \operatorname{Span} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

form invertible matrix: $X = \begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix} \rightsquigarrow X^{-1} = \frac{1}{4} \begin{pmatrix} -1 & 1 \\ 1 & 3 \end{pmatrix}$

diagonalization: $A = XDX^{-1}$ with $D = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$

matrix exponential: $e^{tA} = X \cdot e^{tD} X^{-1} = X \begin{pmatrix} e^{-2t} & 0 \\ 0 & e^{2t} \end{pmatrix} X^{-1}$

$$= \frac{1}{4} \begin{pmatrix} 3e^{-2t} + e^{2t} & -3e^{-2t} + 3e^{2t} \\ -e^{-2t} + e^{2t} & e^{-2t} + 3e^{2t} \end{pmatrix}$$

solution of initial value problem: $\boxed{\begin{matrix} \dot{x} = Ax \\ x(0) = \begin{pmatrix} 0 \\ 4 \end{pmatrix} \end{matrix}} \rightsquigarrow \alpha(t) = e^{tA} \begin{pmatrix} 0 \\ 4 \end{pmatrix}$

$$= \begin{pmatrix} -3e^{-2t} + 3e^{2t} \\ e^{-2t} + 3e^{2t} \end{pmatrix}$$